How can a child learn good ways to stack cups or order objects by height.

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http://www.cs.bham.ac.uk/~axs

Slides here
http://www.cs.bham.ac.uk/research/projects/cogaff/talks/

See also the paper for AISB2010 symposium.
http://www.cs.bham.ac.uk/research/projects/cogaff/10.html#1001

and this discussion of toddler theorems

http://www.cs.bham.ac.uk/research/projects/cogaff/misc/toddler-theorems.html

A new way to do philosophy of mathematics

Most philosophers of mathematics try to understand the nature of mathematical knowledge “from the inside” – starting from features of mathematical knowledge and concepts that all mathematicians (hopefully) would recognise and agree on.

An alternative is to see human mathematics as a product of biological evolution of many species, extended and enriched by development of many individuals using the cognitive mechanisms produced by evolution and the environment.

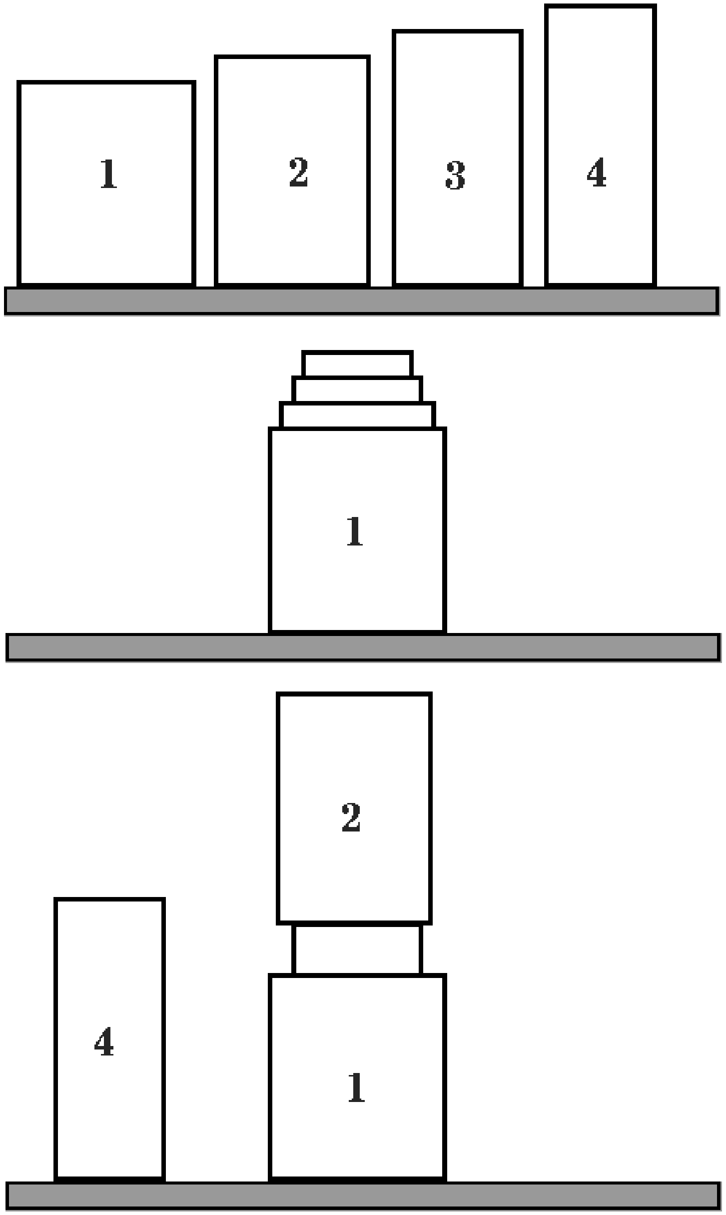
All of that depends on the environment presenting challenges and opportunities, both to evolution and to individuals.

I believe that developing this approach can provide new support for an updated version of Kant’s philosophy of mathematics.

It was the belief about 60 years ago that Kant was basically right about mathematics as I had experienced it (doing a degree in mathematics and physics), whereas most of the philosophers I met (mainly in Oxford) thought Kant’s view had been refuted by Einstein and Hume’s view was the main alternative: namely that mathematical knowledge was non-empirical, as Kant claimed, but essentially “analytic”, or merely a collection of definitional truths and their consequences.

A Process theorem: Toddler Nesting

Understanding how to stack cylinders.



Consider being faced with a collection of cylindrical cans or mugs of different heights and widths all open at the top, as shown in the upper picture.

What do you have to do to get them all tucked together compactly as in the second picture?

What steps should you go through?

A very young child may try putting one mug into another at random, and then get frustrated because situations result like the one depicted at the bottom, where the top mug cannot be pushed down into the mug below it, as required.

By trial and error a child may learn a sequence of actions that works, e.g.

(i) put mug 2 into mug 1
(ii) put mug 4 into mug 3
(iii) lift mug 3 with mug 4 inside it and put both into mug 2

What if there are five mugs, or nine mugs?

Towards a Toddler Nesting Theorem

Later, through trial and error the child may learn to start at one end of the row, and work across the cylinders systematically, putting each one into a wider one next to it, starting with the narrowest.

Another option involves starting at the other end and working systematically.

But what will happen if the cylinders are not initially arranged in increasing order of width?

Eventually a child may discover that there is a simple stacking rule that will work for any initial configuration, always getting the cylinders compactly stacked.

At that stage, or some time later, many children seem to realise that some of the things they have learnt are guaranteed to produce the right result, e.g. no matter how big the mugs are, how they are initially arranged, what they are made of, what colours they are painted, etc., as long as certain conditions are satisfied.

Likewise they may indicate that they realise something is guaranteed to fail: e.g. they may call you silly for trying: are they just using a very high probability estimate?

This is actually a rule about operations that preserve orderings, and can be relied on without testing, as long as the objects involved keep their size and shape neither disappear nor appear out of nowhere. Do **YOU** believe it? If so, why?

Towards a theory of ordering processes

A child may, after a while, notice patterns in the **processes** that occur during attempts to get containers nested, leading eventually to the following discovery somehow represented in the child, though not necessarily in words or a logical formalism:

- If the container that is widest is moved to one side, then repeatedly the widest remaining container is inserted in it, eventually all the containers will be "tucked together compactly" as in the earlier picture.

  A child need not use the word "widest" but may learn to do visual comparisons between containers, and repeatedly discard items after finding something wider.

- Alternatively, starting with the narrowest then repeatedly looking for the narrowest remaining container and inserting the already nested containers into it will achieve the same result (in the reverse order).

  This requires a technique for identifying the narrowest remaining container, which can also be done using visual comparisons.

  The insertion process will be more complex, as it will require the whole nested collection of previously stacked containers to be lifted and inserted into the next container selected.

- A child may discover, perhaps initially by accident, that a way to avoid repeated searches for the biggest or smallest remaining container, is to start by arranging them in a row in order of size.

  Then, whether looking for the smallest or the biggest remaining container, the child need only look at one end of the row.

Later, a child may learn that the stacking process can occur in many different orders.

Knowing how, knowing that, and knowing why

The hypothesis being explored here is that in at least some cases, and the container nesting process might be such a case, the learner acquires different sorts of information in the following order

(though older, more sophisticated learners will learn such things in different orders).

- Through trying various things out a child may learn a way of doing things, e.g. by acquiring chains of associations of the form If the goal is G and the context is C do A

  Compare Nilsson's Teleoreactive programs.

- Later, reflective and communicative competences develop so that the child is able

  (a) to notice that a certain type of goal can be achieved in a particular way,

  (b) to enable others to achieve such goals, by observing them and noticing problems, helping them, correcting them, telling them what to do

  (c) and eventually to formulate one or more strategies that succeed in achieving that type of goal in a variety of contexts.

Still later the child may come to understand why the strategy works.

This may at first involve grasping **perceptual invariants** of the process used, such as:

- If items are repeatedly taken from one end of a row ordered by size, what is left is a row ordered by size
- If items are repeatedly taken from a row ordered by decreasing size then each can be inserted in the previously taken item
- After each insertion the row will shrink, until no item remains in the row.
- The previous process will end with all the items except the first one, inserted, with nothing left to insert.
- These features of the process do not depend on the colour, material, size, weight of the objects, nor whether it is day time night time, or hot or cold in the room: many features of the context are irrelevant.

Understanding why a procedure must work

Young children have to

1. extend their ontology to include new categories of relationship between the cups (e.g. ‘next bigger’, ‘next smaller’)
2. and also so as to develop new categories of process fragment (e.g. ‘insert next smaller’),
3. and ways of composing process fragments (e.g. ‘repeat ... until ...’), including notions of ‘how to start’, ‘stopping condition’, etc.

Doing all that (consciously or not!) can develop

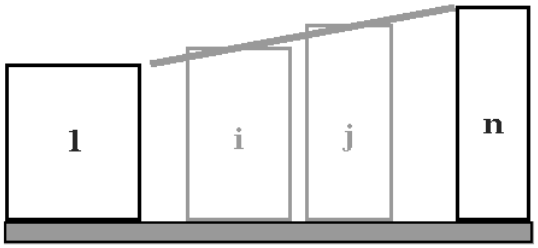
- the competence that enables the task to be completed,
- but not yet the competence to realise that it must always work no matter how many cups there are (under certain conditions).

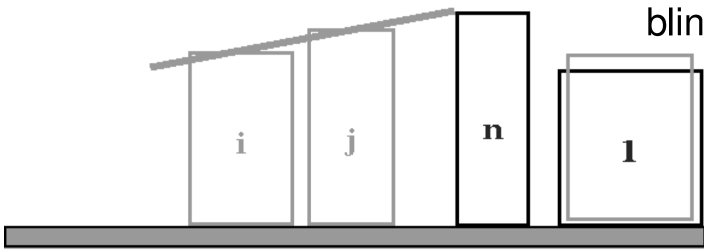
That’s what computer scientists do when ‘proving correctness’.

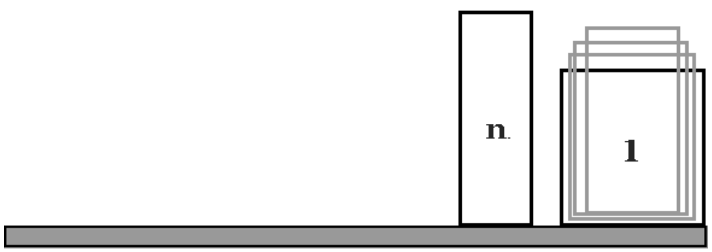
- Of the formulations proposed above, the logical one most clearly allows us to compose a procedure for stacking the cylinders, and then to prove that the procedure achieves the desired goal on the basis of a collection of assumptions about the objects, e.g. they do not divide or merge or move around of their own accord.
- The proof might use logic augmented with the axiom of induction to cover arbitrarily many cylinders to be stacked.
- That sort of proof of correctness using a “Fregean” formalism (in the sense of Sloman 1971) is what theoretical computer scientists mostly look for.
- However, it is unlikely that toddlers can use such forms of representation and reasoning: they turned up very late in human history, and normally turn up quite late in individual human development.
- Is there another way of thinking about the process that might allow a young child to grasp that it will necessarily achieve the goal, under certain conditions?

A more “intuitive” mode of reasoning?

Children (and most adults) apparently cannot construct logical proofs, but I conjecture that they can grasp the necessity of the outcome by using a different form of representation based on “analogical representations” (Sloman 1971, Peirce’s Icons?):







CONJECTURE:

various stages of the process are represented in a manner that is closer to how they are perceived visually than how they are described logically.

(‘Perceived visually’ has to be qualified for people blind from birth. But they still benefit from having had ancestors with vision if the blindness is caused by a peripheral defect not something central.)

The reasoning is easiest in the special case where the containers are first all arranged in a row, decreasing in width.

That configuration can be thought of schematically as having a widest cup on the left and a thinnest one on the right, with every adjacent pair (i,j) having the wider (i) on the left, but without representing specifically how many cups there are, nor how big any of them is, nor exactly where they are:
This requires a “generic” or “schematic” analogical representation – using “visual ellipsis”.

(Cf Kant vs. Berkeley, on generic triangle images.)

From that starting configuration (top image) you can envisage a first step of moving the widest cup to a new location (middle image) and then repeatedly taking the last cup on the left and putting it in the growing stack.

So there will be a generic situation that can be visualised, with the row of remaining cups, decreasing in width from left to right, and the already stacked cups somewhere separate – until one cup is left, and finally put inside the stack.

Representing a process visually

The crucial discovery is that there is an invariant structure while the repeated operations are performed when there are two or more containers left from the original row.

This can be represented by a generic process step that moves the leftmost cup from the row and puts it inside the last moved cup in the stack.

During this process there is still

- a row of remaining cups ordered by decreasing width from left to right
- Set of already stacked cups
- until there's only one cup left in the row, and putting that one in achieves the goal.

That's an invariant that can also be expressed logically, but probably not by a toddler!

I know some humans can do all this, but I don't yet know how to build AI systems with the appropriate forms of representation, self-monitoring capabilities, architecture, etc.

This process is similar in character to a proof in Euclidean geometry where a diagram (on paper or in the head) indicates the possibility of transforming one configuration to another, e.g. by adding construction lines, or by marking lines or angles equal, etc. (Sloman 1971)

Each such diagram, and each transformation, represents not just one particular case but a whole class of cases: the ontology of structures and processes is partly diagrammatically characterised. (How does that work?)

A more formal theory of container nesting

Some learners will develop mathematical ways of thinking, using mathematical formalisms and strategies, and thereby gain more general understanding of why something works.

For example, the learner may come to realise that the previously learnt successful strategies are special cases of more general strategies.

For example, stacking the nesting cylinders need not work monotonically from smallest to largest or from largest to smallest, because of the following facts about orderings of things and actions:

1. If n objects have different sizes and no two are the same size, then there is a decreasing-size-ordered sequence: O_1, O_2, O_3, ... O_n, where any two objects O_i, O_j, will be in one of two relations, $Bigger(O_i, O_j)$, or $Bigger(O_j, O_i)$, and each except O_1 has exactly one immediate predecessor and each except O_n has exactly one immediate successor: the sequence is “well-ordered” by size.
   (Defining “immediate predecessor” and “immediate successor” is left as an exercise.)
2. The sequence exists independently of the physical arrangement of the objects.
3. For any object O_i, other than O_1, its immediate predecessor can be inserted into it, as follows:
   for every other object O_j
   - if $Bigger(O_j, O_i)$ then reject O_j (it’s too big)
   - if Ok is found such that $Bigger(O_i, Ok) \& Bigger(Ok, O_j)$ then reject O_j (not immediate successor)
   - Otherwise insert O_j in O_i.
4. If n-1 actions are performed, where each action involves putting an object in its immediate predecessor (the next bigger), then the result will be a rearrangement of the objects so that each of the objects O_i, except for O_n, contains all the objects O_j such that $Bigger(O_i, O_j)$.

This formulation allows different orderings of the actions, but step 3 is slow and error prone.

To be revised/extended

See additional presentations on toddler theorems here:

`http://www.cs.bham.ac.uk/research/projects/cogaff/talks/#toddler`