FUNCTIONS AND ROGATORS

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Abstract Published Separately

Frege, and others, have made extensive use of the notion of a function, for example in analysing the role of quantification, the notion of a function being defined, usually, in the manner familiar to mathematicians, and illustrated with mathematical examples. On this view functions satisfy extensional criteria for identity. It is not usually noticed that in non-mathematical contexts the things which are thought of as analogous to functions are, in certain respects, unlike the functions of mathematics. These differences provide a reason for saying that there are entities, analogous to functions, but which do not satisfy extensional criteria for identity. For example, if we take the supposed function 'x is red' and consider its value (truth or falsity) for some such argument as the lamp post nearest my front door, then we see that what the value is depends not only on which object is taken as argument, and the 'function', but also on contingent facts about the object, in particular, what colour it happens to have. Even if the lamp post is red (and the value is truth), the same lamp post might have been green, if it had been painted differently. So it looks as if we need something like a function, but not extensional, of which we can say that it might have had a value different from that which it does have. We cannot say this of a function considered simply as a set of ordered pairs, for if the same argument had had a different value it would not have been the same function. These non-extensional entities are described as 'rogators', and the paper is concerned to explain what the function-rogator distinction is, how it differs from certain other distinctions, and to illustrate its importance in logic, from the philosophical point of view.

FUNCTIONS AND ROGATORS[1])

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Section A

1. The concept of a "function", though frequently *used* by logicians (e.g. in talking about truth-functions or propositional functions), has rarely been *discussed* systematically since the writings of Frege and Russell. The notion may be approached in several different ways, either syntactically, through the notion of a "function-sign", or semantically, through the notion of what corresponds to such signs. The syntactical approach may either deal with function-signs as "incomplete" or "unsaturated", following Frege, or it may deal with them as complete signs (e.g. signs containing variable-letters "*x*", "*y*" etc., or signs prefixed with Church's lambda operator). The latter approach is more common, the former more fundamental. (A similar distinction could be made at the semantic level: see end of par. 11, below.) The semantic approaches may also be subdivided into two sorts, depending on whether they are intensional or extensional. Once again, the latter is more common, the former more fundamental. It would be of considerable interest and importance for the philosophy of logic to analyse these various approaches and their interrelations, especially as most modern text-books are somewhat narrow, favouring one or other approach as the only acceptable one, others being, at best, mentioned with a few disparaging remarks.[2])

[1]) I wish to thank Michael Dummett and members of the Philosophy department at Hull University, for helpful comment and criticism at various stages in the development of this paper.

[2]) See, for example, P. Suppes, Introduction to Logic (Princeton 1957) 229f. Similar remarks are made by A. Tarski on p. 72 of his Introduction to Logic (New York 1946).

2. In this paper only the semantic conceptions will be discussed. An attempt will be made to explain the difference between the extensional and the intensional approach, with the aim of showing that the latter is not just a confused version of the former, but is something quite different, and, in one sense, prior to the other. This is not a new suggestion. Some of what I have to say has been said before, e.g. by Russell (in *Introduction to Mathematical Philosophy*, p. 12 ff, p. 183 ff) and F. P. Ramsey (in *The Foundations of Mathematics*, p. 15). But I am not aware of the existence of any *detailed* discussion of the distinction or its applications. This first section will be devoted to a brief explanation of the distinction. The next (section B) will compare it with other distinctions likely to be confused with it. In the final section (C) some applications will be mentioned.

3. The following notion of a function-sign will be assumed to be familiar: a function-sign is obtained from a sentence or referring expression (for example), by replacing one or more words or phrases in it by so-called "variable-letters" such as "*x*" or "*y*". Examples are:

"The mother of *x*",
"The town in which *x* was born",
"*y* is the father of *x*".

The semantic approach involves regarding a function as something which, in some sense, corresponds to such a function-sign. It is said to take *arguments* and yield *values* correlated with the arguments. If a name or sign for an argument is substituted for each variable-letter in a function-sign, then the result is taken to be a name or sign for the value correlated by the corresponding function with that argument or set of arguments. The things which correspond to function-signs and which take arguments and yield values are normally described as "*functions*", but I shall use two words "function" and "rogator" to mark the difference between the extensional and the intensional concepts. (This is less cumbersome than talking about "extensional functions" and "intensional functions", and avoids confusion which might arise out of the fact that this latter terminology has been used to mark another distinction, to be mentioned below. I retain the word "function" for the extensional concept, since that seems to be its normal use at present, though it could be argued that the normal meaning is somewhat indefinite in this respect. A mathematician recently said to me that he thought of a function as a sort of machine, which churned out numbers as numbers were fed into it. This could be taken as an intensional explanation.) Functions and rogators, then, are thought of as corresponding to function-signs, and as taking arguments and yielding values. This much they have in common.

4. In order to explain the difference between functions and rogators we need the notion of "extensional equivalence". Two functions, or two rogators, are said to be extensionally equivalent if (a) they each have values for the same arguments, and (b) they correlate the same values with the same arguments. That is, two functions, or rogators, "Fx" and "Gx" are extensionally equivalent if, and only if,

$$(x)\,(y)\,[(y = Fx) \equiv (y = Gx)].$$

The difference can now be explained. Extensional equivalence is a necessary and sufficient condition for the identity of functions, but *not* for identity of rogators. Thus, the functions corresponding to "the mother of x" and "the woman first loved by x" may be extensionally equivalent, in which case they will be one and the same function. But this does not mean that there will be one and the same rogator corresponding to them, even though the two rogators are extensionally equivalent. To say that rogators are intensional entities, then, is simply to say that extensional equivalence does not guarantee identity of rogators: there are no further metaphysical or psychological implications. Of course, this account of the difference between rogators and functions is not a definition of either. We may regard it as a partial definition, or a criterion for adequacy of a definition of the notions. Let us now see if we can give adequate complete definitions.

5. If we are allowed to make use of the notion of a set, then we can define "function" in the familiar way as a set of ordered pairs satisfying the condition that no two pairs in the set have the same second element. Since sets satisfy extensional criteria for identity it follows that this definition of "function" fits the criterion of the previous paragraph. That is, if two functions contain exactly the same ordered pairs, then they are identical, since the sets of ordered pairs are identical. But containing exactly the same ordered pairs means correlating the same values with the same arguments. So we have found something (the usual thing) that can be called a "function".

6. It is not so easy to give a full definition of "rogator". We want things which take arguments and correlate them with values, thereby generating sets of ordered pairs, but which do not satisfy the extensional criterion for identity. I claim that we are talking about such things whenever we talk about functions as pairing off elements according to a *plan*[1]), or mention the *way* in which a function yields or produces its value from its argument[2]) or the *principle* of classification[3]). For example, it is clear that to the two expressions

(1) "The sum of the first x odd numbers" and
(2) "The number which is equal to the product of x by itself"

there correspond two different *methods* or *principles* of calculation, even though when applied to positive integers they always give the same result. So although the two *functions* (on the domain of positive integers) corresponding to (1) and (2) are identical, there are other things which are not. Hence these other things, namely the rules, methods or principles (etc.) do not satisfy extensional criteria for identity. Let us therefore say that in talking about rogators we are simply talking about these other things, in effect, and that the criteria for identity of rogators are simply the criteria which we normally use for identifying and distinguishing these other things. Then talking about an object as an argument for a rogator which correlates it with a value, is just a neater, and more general, way of talking about the object as something to which a rule or principle may be applied in order to yield a result or outcome of the application. I shall not attempt to give an explicit definition of "rogator" in terms of "rule" or "method" of "principle", etc., since (a) these terms are in some contexts subject to the extensional-intensional ambiguity themselves, (b) it is not clear that their use is sufficiently general and (c) it would be odd to describe them as having arguments and values. The connection between the concept of a "rogator" and these other concepts will simply have to

[1]) W. V. O. Quine, Mathematical Logic (Cambridge, U.S.A. 1955) 198.
[2]) A. Church, Introduction to Mathematical Logic (Princeton 1956) 16.
[3]) F. P. Ramsey, The Foundations of Mathematics (London 1931) 15.

be hinted at and illustrated by the remarks already made, and the examples which will now be discussed.

7. We have admitted that the functions "the woman first loved by x" and "the mother of x" may be identical. But even if they are, it is clear that the principle by which we pick out a woman as someone's mother is quite different from the principle by which we select a woman as the first person loved by that person, even if we end up with the same woman in each case. So here we are applying non-extensional criteria for identity of the principles involved, and these criteria enable us to distinguish two rogators, even if there is only one function. Again, if we consider the expression "the town in which x was born", then we may say that there is a function corresponding to it which correlates (some, but not all) persons with towns. Suppose that Aristotle's first pupil, whoever he was, was born in Athens. Then Athens is the value of the function for that man as argument. But that man might have been born elsewhere, for example if his mother had decided to go on holiday just before his birth. In that case a different town would have been the value for the same man as argument. But a value of *what*? A different town could not be the value of the same *function*, for then the set of ordered pairs would be different, and so, since a function just is a set of ordered pairs (or at any rate something satisfying extensional criteria for identity), it would be a different function. Hence, if, as seems quite natural, we wish to say that the same something might have had a different value for the same argument, then, if we are not to contradict ourselves, we must regard the "something" as not satisfying extensional criteria for identity. Clearly, it is the same *rogator* that is wanted: for one and the same principle or rule might have correlated the same man with a different town if he had been born not in Athens but elsewhere. That is, the rogator corresponding to that rule might have had a different value for the same argument.

8. It is important to note that the remarks made in the previous paragraph could not have been made, and the reader would not have understood them, if they had not employed the concept of a rogator or some other non-extensional concept. We can therefore take the fact that the remarks are intelligible as demonstrating that there are such things as rogators, or at least that the concept of a rogator is a coherent one, and not unfamiliar. There is a further argument, used by Russell, to show that there must be rogators. The argument is simply that unless there were such things we should not be able to talk about individual functions such as "the square of x" or "the town in which x was born", whose domains are either infinite or unsurveyable on account of being scattered about in space and time. For how can I have this set of ordered pairs in mind rather than that, and how can I know that you and I are talking about the same set in these cases? The function "the square of x" contains infinitely many different pairs of numbers, and the other function includes pairs containing persons and towns that I have never seen or heard of (especially if it applies to persons who have lived in the past, or will live in the future). So in neither case can I say that I have in mind just this function because I have identified all the pairs in it. And I cannot say that I am sure you have the same function in mind on the basis of having checked through the set of pairs which you have in mind. Thus, if there is one function that I have in mind, and if you have the same function in mind, it can only be because we use some principle, or rule, i.e. a rogator, according to which we can tell whether an ordered pair does or does not belong to the function in question. It follows that since we can and do identify and talk about functions with infinite or unsurveyed domains, there are such things as rogators. I am not saying that extensional functions could not *exist* if there were no rogators, only that individual ones could not be *talked about* or even *thought about* without them. (Though as pointed out by Ramsey in *The Foundations of Mathematics*, pp. 15 and 22, it may be possible to make *general* assertions about them, not mentioning individual ones, without presupposing the existence of a rogator. Whether there are some functions – or sets – to which no rogators correspond, so that they cannot be talked about or thought about individually, is a question which I shall not discuss. One form of Platonism involves giving an affirmative answer to this question. This sort of view seems to have lain behind the axiom of reducibility, and Ramsey's claim that the axiom was unnecessary.)

9. These considerations seem to establish that there are such things as rogators and that they are, in one sense at least (namely, epistemologically), prior to functions. Although this fact was acknowledged by Russell, he did not wish to pay much attention to it, since he was preoccupied with giving mathematics a logical foundation, and apparently thought this could be done without introducing intensional considerations. (See *Introduction to Mathematical Philosophy*, p. 187.) This may also explain why Frege apparently was not very interested in an intensional approach to the concept of a function. It should be noted at this stage that, although I have indicated in a rough sort of way what sorts of things rogators are, I have not yet given a *definition*, for I have not yet stated a set of necessary and sufficient criteria for identity of rogators. Certainly if "*Fx*" and "*Gx*" are the same rogator (involve the same rule or principle) then they must correlate the same arguments with the same values, and if a certain argument (e.g. Aristotle's first pupil) would have been correlated with a different value by "*Fx*" (e.g. "the town in which *x* was born") if the world had been different, then in the same conditions "*Gx*" would have had the same value for that argument. So extensional equivalence in all possible states of affairs (i.e. necessary extensional equivalence) is a necessary condition for identity of rogators. But we do not wish to say that it is a *sufficient* condition, since we wish to say that the two rogators mentioned in par. 6, namely "the sum of the first *x* odd numbers" and "the number which is equal to the product of *x* by itself" (defined on the domain of positive integers), are different rogators, despite the fact that they necessarily, i.e. in all possible states of the world, have the same values for the same arguments. It might be thought that the only difference is that they correspond to different *signs*, that is that the criteria for identity of rogators are partly syntactical. But this is not so, for it is possible that in some strange language the expression "the sum of the first five odd numbers" means what *we* mean by "the number which is equal to the product of five by itself", in which case the rogator corresponding to their expression "the sum of the first *x* odd numbers" would be different from the rogator corresponding to ours, since it would correspond to a different principle of calculation, despite the syntactical and extensional equivalence. These remarks should suggest that it is not easy to give necessary and sufficient criteria for identity of rogators, i.e. to explain, in a clear and non-circular manner, how we identify and discriminate rules or principles or methods of calculation. Ultimately, we simply have to make use of something like the notions "same pattern" and "different pattern", i.e. the notions of identity and difference of properties or universals. All explanation of meanings must start with examples, and it seems clear that we have here something which can be taught by means of examples, but which cannot be described, except in a circular manner. I shall therefore not attempt to formulate sufficient criteria for identity of rogators.

10. Normally, when we wish to talk about a function, we specify the one in question not by enumerating its arguments and values, but by indicating some *principle* according to which they can be picked out. And this is usually adequately achieved by the use of a function-sign as illustrated in par. 3, above, for if the function-sign is constructed out of parts which have unambiguous meanings, the method of construction, together with those meanings, uniquely determines a principle or rogator. This permits us to talk about the rogator corresponding to such a sign, just as we talk about the function corresponding to it. We could, of course, introduce a notation for talking about functions and rogators by enclosing the function-sign in different sorts of quotation marks or by using prefixes, such as Church's prefix "λx-" for functions, and perhaps "ρx-" for rogators. However, if we talk about "the function '*Fx*'" or "the rogator '*Fx*'" there should be no ambiguity. (In such locutions the letter "*x*" is, of course, a sort of bound variable.) To one function there generally correspond many different rogators, since one and the same set of ordered pairs may be picked out in many different ways, i.e. according to many different rules or principles. Since, for reasons mentioned, no complete definition of "rogator" has been given, it may be useful to compare and contrast the function/rogator distinction with several other distinctions with which some may be inclined to confuse it.

Section B

11. Near the end of section 71 of *The Logical Syntax of Language* Carnap implies that Frege's distinction between a function and its value-range (*Wertverlauf*) is a distinction between intensional and extensional entities. But this seems to be a misunderstanding, for this distinction of Frege's is a distinction between entities which are "complete" and entities which are "incomplete" or "unsaturated", and, as far as I can see, has nothing to do with different criteria for identity. Frege did not use "function" to mean "set of ordered pairs", since he defined the notion of a set or class *in terms of* the notion of a function. Nevertheless, it seems likely that he thought of functions in an extensional way, since he thought of concepts as being functions of a certain sort, and he thought of them as extensional. For he wrote: "coincidence in extension is a necessary and sufficient criterion for the occurrence between concepts of the relation corresponding to identity between objects". (See *Translations from the Philosophical Writings of Gottlob Frege*, by Geach and Black, p. 80, and also "Class and Concept" by P. T. Geach, in Philosophical Review, October 1955.) Strictly speaking, Frege could not regard the relation of identity as applicable to functions, since, for him, they were "incomplete" or "unsaturated", and this was why he had to introduce value-ranges. (Loc. cit. pp. 26ff.) Frege's distinction between complete and incomplete entities seems to be based, in the first place, on a syntactical distinction between function-signs and argument-signs. (Loc. cit. pp. 12ff., 32, 113ff, 152.) He apparently thought that the analysis of (say) a sentence into argument-sign and function-sign could be parallelled by analysis of *what was expressed* into function and argument, or, in some cases, concept and object. But the important thing about his functions, or concepts, was not intensionality but incompleteness. So Frege's distinction was not the same as the function/rogator distinction. Indeed, a follower of Frege might argue that just as Frege distinguished between "incomplete" functions and "complete" value-ranges, so ought I to distinguish between "incomplete" and "complete" sorts of rogators. The incomplete ones would correspond to Frege's incomplete function-signs, such as "the mother of . .", whereas the complete ones would correspond to complete signs or names for rogators, such as "the rogator mentioned in the previous sentence". So Frege's distinction cuts across mine.

12. Next it may be thought that the notion of a rogator might be explained in terms of the notion of a function by saying that if R is the rogator corresponding to the function-sign "Fx", and F is the corresponding function, then R is just a function which takes different arguments and values from F, as follows. If any object is taken as an argument of F, then that argument must be picked out or identified in some way, and the method by which it is picked out will fix the sense of the argument-sign which refers to it. Similarly, any sign which picks out the value of

F for a given argument must have a sense, corresponding to the way in which the thing is picked out. The suggestion I am considering is that R is just a function from senses to senses: that is, instead of taking objects as arguments and values, it takes *senses* of argument-signs and correlates them with *senses* of signs for the corresponding values of F. So R is supposed to be a set of ordered pairs of senses of signs, or a set of ordered pairs of ways of identifying arguments and values. Now there is no reason at all why we should not talk about such functions from senses to senses,and it seems certain that they would mirror some of the properties of rogators, such as there being many different rogators corresponding to one function. But the argument of par. 8 shows clearly that such "second-level" functions will not do everything that rogators can do. In particular they cannot explain how we are able to think and talk about particular functions whose arguments and values we cannot enumerate. For, if there are too many arguments, then there will automatically be too many senses of possible argument-signs, or ways of identifying objects, since every one of the arguments may be referred to in many different ways. Hence, if F_2 is a second-level function from senses to senses, corresponding to the "unsurveyable" function F, then F_2 will be even more unsurveyable, and we shall need a rogator in order to talk about *it*! This argument is most important, for it can be used against any attempt to construe a rogator as a kind of function. For example, Professor Richard Montague, referring to work done by Tarski, suggested to me that we could avoid talking about rogators if we talked instead about functions with an additional argument-place, to be filled by a (sign for a) possible state of the world, which would certainly enable us to deal with the examples of par. 7. But if such functions were really extensional, that is, if they consisted of sets of ordered triples, one member of each triple being a (sign for a) possible state of the world, then, as before, every such function would be far more complicated than the set of ordered pairs corresponding to the actual state of the world. For example, since we cannot enumerate arguments and values for the function "the town in which x was born", we shall find it even more difficult, on account of the geater multiplicity of arguments, to enumerate arguments and values for the function "the town in which x was (or would have been) born in possible world y" (apart from any difficulties in identifying the same particulars in all possible states of the world). Hence, as before, if we are to think or talk about such a function, we need something non-extensional, such as a principle of correlation, or a rogator, by means of which it can be identified. This sort of argument works equally well against the much cruder suggestion that a rogator is just a time-dependent function. I shall not elaborate on this suggestion, for it should be clear by now that a rogator is not a type of function at all.

13. The terminology of "intensional functions" and "extensional functions" has been used by Russell (*Principia Mathematica*, 2nd ed., p. 72ff) and Kneale (*The Development of Logic*, p. 609), but for them these terms are not so much concerned with the distinction between rogators and functions as with a different distinction, noticed by Frege (See "On Sense and Reference", in *Translations*). Quine has illuminatingly described the distinction as being between "referentially opaque" and "referentially transparent" contexts. The distinction may be illustrated by the pair of function-signs:

(1) "the day on which our chairman first thought about x" and
(2) "the day on which our chairman was first seen by x".

These both look as if they correspond to functions in the usual way, but there is a difference: for if a sign which does not refer to anything is substitued for "x" in (2), then the resulting sign does not refer to anything, and if two signs referring to the same argument are substituted in (2), then the two resulting complex expressions cannot refer to different days. On the other hand, there is no person referred to by "Mr. Pickwick", yet if it is substituted in (1) the resulting expression will probably refer to a definite day, and if the two expressions "Bertrand Russell" and "The author of *The Principles of Mathematics*", which refer to the same person, are substituted in turn for "x" in (1), then it is very likely that the resulting expressions will pick out different days. In short, the value of (2) depends only on what, if anything, is taken as argument, whereas the value of (1) seems to depend on how the argument is identified, that is, on the sense of the sign for the argument. We may say that (1) corresponds to an "oblique" function, (2) to a "direct" function. But it should not be thought that this distinction is the same as the distinction between rogators and functions. For the rogator "the town in which x was born" takes a value only if a sign which refers to something is taken as argument-sign: it cannot have a value for a non-existent argument. Moreover, its value depends only on which person or animal is the argument, not on how the argument is identified. It is in order to avoid this ambiguity that I refrained from describing rogators as "intensional functions": as already remarked, such a terminology might be confused with Russell's.

14. Finally, it is clear that the distinction between rogator and function is in some ways analogous to Frege's distinction between sense and reference (op. cit.). For the reference of a name (or definite description) is an object, and the sense is that in virtue of which this object is the one referred to by the name or other expression. Similarly, it should by now be clear that the *rogator* corresponding to a functional expression is that in virtue of which a particular *function* (set of ordered pairs) is the one corresponding to that expression. But, for Frege's purposes, it is important to distinguish between a complete expression referring to a function, e.g. the expression "the function described in paragraph 7", or "the function (corresponding to) 'the square of x'", and an incomplete function-sign, such as that which is common to "the square of six" and "the square of twenty-two". Strictly speaking, the latter is not a sign at all, but an aspect or pattern or structure common to different signs. We could say that rogators and functions serve respectively as senses and referents for such "incomplete" entities. The previous kind, being "complete" signs, already fall under Frege's discussion of sense and reference. Despite the analogies, to identify rogators with senses of signs involves some linguistic strain, since it is odd to say that a sense can take arguments and have values. (I do not know whether Frege himself made any attempt to extend his sense/reference distinction to what he called function-signs.) This completes the comparison of the rogator/function distinction with other distinctions, and now all that remains to be done is to describe some applications of the distinction.

Section C

15. The first and most obvious application is analogous to Frege's application of the sense/reference distinction to identity statements. For, just as identity statements, such as "The evening star is (identical with) the morning star" would be either quite trivially true or self-contradictory if referring expressions were directly associated with objects without the mediation of a *sense* (or method of identification), so also would statements of extensional equivalence between functions, such as

> "For any argument *x*, the function 'the mother of *x*' has the same value as the function 'the first woman loved by *x*' ",

reduce either to mere triviality or to self-contradiction if the sign for a function were directly correlated with a set of ordered pairs without the mediation of a *rogator* (that is, a rule or principle). The significance of statements of identity depends on the fact that it may be a significant (e.g. contingent) question whether two senses pick out the same referent. Similarly, it is because the question whether two rogators pick out the same function, the same set of ordered pairs, may be a significant (e.g. empirical) question, that statements of extensional equivalence have any significance.

16. Secondly, once the distinction has been made, we can see that the notion of a function can be explained or analysed or "reduced" in terms of the notions of a rogator and extensional equivalence. But it is not possible to "reduce" the notion of a rogator to that of a function, or set. A third application may be mentioned briefly here, in connection with this. Since "function" can be defined in terms of "rogator", and since a rogator is something like a rule or principle, which can be identified independently of any enumeration of the objects which it correlates, it follows that there is something wrong with the statement in *Principia Mathematica* (2nd ed., p. 39) that a function is only well-defined if its values are already well-defined. So there is something wrong with *one* argument in favour of the vicious circle principle. I shall not enlarge on this, but it seems likely that further investigation might lead to a better understanding of some of the problems connected with the ramified theory of types and the axiom of reducibility.

17. The fourth application which I shall mention is one which seems to me to be particularly interesting and important for the philosophy of logic. If we look back at two of our examples of rogators, namely "the town in which *x* was born", and "the square of *x*", which may be referred to as "*Fx*" and "*Gx*" respectively, we notice the following difference (cf. par. 7, above): suppose the value of "*Fx*" for Aristotle's first pupil as argument to be the town Athens. Then the same rogator might have had a different value for the same argument, since the man in question might have been born in some other town. However, if we take the number six as an argument for the second rogator, we see that its value is thirty six, and could not have been anything else in any circumstances. It looks as if we have a distinction between two sorts of rogators: one sort has a value which depends on how things happen to be in the world, whereas the other fully determines its value independently of contingent facts. As pointed out to me by Mr. Dummett, there is something odd about putting the distinction in this way, since if we take different argument-signs, the rogators in question seem to exchange their positions with regard to the distinction. For example, if we apply "*Fx*" to the argument identified as "the man whose mother was the first woman in 1930 to give birth in Rome to her only son", then it is clear that the value (if there is one at all) *must* be Rome. On the other hand if we apply the rogator "*Gx*" to the argument identified as "the number of hours between lunch and dinner according to the Colloquium time-table", then it seems that although the value is thirty six, it makes good sense to say that it might have been different, if the printers had made a mistake on the time-table, or if the eating arrangements had been different. Moreover, a problem arises if we apply several different rogators, such as "the mother of *x*", "the father of *x*", "the wife of *x*", "the day on which *x* was born" etc., to the person taken previously as argument for "*Fx*", namely Aristotle's first pupil. For even if it makes sense to say of each of these in turn that it might have had a different value for the same argument, it certainly does not make sense to say that all of them might simultaneously have had different values for the same argument. For how could one have had a different mother, a different father, a different wife, been born on a different day, and in a different town, etc., and still been the same person?

18. Such difficulties are avoided if we describe the contrast not as one between types of rogators, but as a contrast between cases of application of a rogator to an argument to yield a value. In general, the value of a rogator for a given argument is fully determined by three factors

(a) the rogator itself (i.e. the principle according to which arguments are correlated with values), (b) the method by which the argument is identified (or, in particular, the sense of the expression taken as argument-sign) and (c) contingent facts, or how things happen to be in the world. As the application of "*Fx*" to Aristotle's first pupil, and the application of "*Gx*" to the number of hours between lunch and dinner according to the Colloquium time-table show, it is not generally the case that *two* of the factors suffice to determine a value. On the other hand, some of the other examples show that in some cases the first two factors (a) and (b) do suffice. Thus, how things happen to be in the world cannot affect the outcome of applying the rogator "the square of x" to an argument identified as the number six.

19. We can now give a precise formulation of the distinction referred to two paragraphs ago. It is a distinction between cases where two (or one) of the factors (a) (b) and (c) suffice to determine the value of a rogator for an argument identified in a certain way, and cases where all three factors are required. In particular, when the third factor, how things happen to be in the world, is not relevant, i.e. where (a) and (b) suffice to determine the value, I shall say that the application of the rogator satisfies the *NCD-condition* (the non-contingent determination condition). In most mathematical contexts the NCD-condition is satisfied, since the standard methods of identifying numbers, or other mathematical objects (e.g. as things which satisfy certain axioms), are such that once they have been used to fix a number they automatically determine all its properties and relations to other numbers, and therefore also help to determine the values of mathematical rogators taking those numbers as arguments. On the other hand, the normal methods of identifying non-mathematical objects, such as persons, places, etc., do not automatically determine their properties and their relations to other objects of the same kind, in general: these depend on contingent facts. Since the NCD-condition is normally satisfied in mathematical contexts, philosophers primarily concerned with the foundations of mathematics have not felt any pressing need to take account of the distinction between cases in which it is satisfied and cases in which it is not. This is connected with the fact that the distinction cannot be made if the concept of a *function* is used instead of the concept of a *rogator*. For, since a function is identified in terms of which objects it correlates with which (i.e. *via* a set of ordered pairs), it makes no sense to distinguish cases in which a function might have had a different value from cases in which it could not have had a different value (for the same argument). For, since functions are extensional, the value cannot be different unless the function is.

20. This shows that the concept of a rogator or some other non-extensional concept is essential for making the distinction between cases where the NCD-condition is satisfied and cases where it is not. It may be noted that there is something unsatisfactory about describing the distinction in terms of factors which determine the value of a rogator. For it might be said that in all cases the value is in some sense determined by the two factors (a) the rogator and (b) the sense of the argument-sign, the difference between the general case and cases where the NCD-condition is satisfied being that in the latter the value is determined in two different ways. (E.g. if "*a*" is an expression referring to a person, then the sign "the town in which *a* was born" usually identifies a town. On the other hand, if "*a*" is the expression "the man whose mother was the first woman in 1930 to give birth in Rome to her only son", then we have two different ways of referring to the value, the new one being by means of the word "Rome". These two expressions – or their senses – must, independently of contingent facts, pick out the same thing, and this, it might be said, is all that satisfaction of the NCD-condition comes to.) This way of looking at the distinction, though illuminating, makes no difference for our present purposes and will not be discussed any further. It should also be noted that I have not taken account of the fact that in some cases, even where the NCD-condition appears to be satisfied, the three factors (a), (b) and (c) may fail to determine a value at all, on account of the failure of some term to refer, or for some other reason. E.g. Aristotle might not have had *one* first pupil, if at the start he took his pupils in groups; or his first pupil, if there was one, may not have been born in any town at all. In either case, applying the rogator "the town in which *x* was born" to Aristotle's first pupil could yield no value. This may be allowed for by inserting the qualification "if it has a value" at various points in the discussion. It has been omitted in the interests of simplicity.

21. We have seen how the notion of a rogator, unlike the notion of a function, can be used in a formulation of the distinction between satisfaction of the NCD-condition and non-satisfaction of the condition. This may now be illustrated and applied further. Any two-valued rogator can be used to define a propositional function. If $R(x, y, z, \ldots)$ is the rogator, whose value for any set of arguments is always one or other of the two objects K and L, whatever they may be, then there corresponds to it a propositional function which is satisfied by the ordered set of objects $(a, b, c, \ldots)$ if and only if $R(x, y, z, \ldots)$ has the value K for these objects as arguments (and a similar propositional function may be defined in terms of L). Conversely, it is possible to think of any propositional function as if it were simply the "value-range" (in Frege's sense) of a rogator taking the words "true" and "false", or any other arbitrarily selected pair of objects, as values.[1]) The normal methods of replacing non-logical words and phrases in a sentence by variables to yield a sentential matrix can be used to represent such a rogator: e.g. "*x* is *A*", "All *A*'s except *x* and *y* are *B*'s", "*p* or *q*", "*q* or not-*p*", etc., can all be thought of as representing what I call *propositional rogators*. In general, the logical form of a proposition can always be thought of as a rogator, sometimes a rogator whose arguments are of different types, as in "*x* is *A*". This shows that the familiar analysis of propositions in terms of functions and arguments can be replaced by an analysis in terms of rogators and arguments. The sense of a sentence expressing a proposition is then partly determined by the rogator corresponding to the logical words and constructions in the sentence. We can conclude that insofar as rogators are prior to functions (i.e. to sets of ordered pairs), the sense of a proposition is prior to the set of its truth-conditions. (This might be

[1]) This seems to be what is important in Frege's decision to regard sentences as names of truth-values. To object that this is an unacceptable use of the word "name" is to miss the important point. The main advantage of this move is that it yields a theory of meanings, propositions and truth which fully accounts for all the properties and relations of these concepts which are of interest to logicians, without depending on discussions of such notions as "thinking", "asserting", "communicating" or the presuppositions and implications of such activities as statement-making. In short, it clearly sorts out confusions between logic and the sociology or psychology of language. In his paper on "Truth" (Proc. Aristotelian Society 1958–59) Michael Dummett attempts to criticise such a Fregean theory, but I think it can be shown that his criticisms fail to take account of its full potentialities. Perhaps Frege was not aware of them either. (It is hoped that this will be developed in another paper.)

developed to support a claim that there is a sense in which meaning is prior to use.)

22. We have seen that in most non-mathematical contexts the NCD-condition is not satisfied by the application of rogators to arguments, and this applies equally to the propositional rogators corresponding to logical constants or logical forms. For example, the rogator "*p* or *q*" may be applied to the two propositions "the moon is shining" and "dawn is breaking", and its value will be (say) the word "true" or the word "false". But there is no way of finding out *which* it is, even if the time and place of utterance are known, except by empirical investigation of contingent facts, for the value is not fully determined by the rogator and the methods by which the arguments are identified: the NCD-condition is not satisfied. This fact, that in general the third factor (c), mentioned in par. 18, is relevant to the value of a propositional rogator is what justifies correspondence theories of truth. To say that truth is a matter of correspondence with facts, is to mention one instance of the generalisation that the value of a rogator depends on how things happen to be in the world. (This shows that falsity is also a matter of correspondence with the facts.) Similarly, to say that any proposition determines a set of possible states of the world in which it would be true, its "truth-conditions", is to draw attention to one application of the more general fact that if $R(x, y, z, \ldots)$ is a rogator, $(a, b, c, \ldots)$ arguments of R, and K a possible value of R, then the rogator, the argument-set and the value K together determine a set of possible states of the world, namely those in which R would take the value K for the arguments in the set $(a, b, c, \ldots)$. By considering rogators which take more than two values we thus find a natural interpretation for systems of many-valued logic. The fact that the propositional rogator "*p* or *q*", and the methods for identification of its arguments, do not in general suffice without the third factor to determine the value of the rogator, is what makes it possible for such logical words and constructions to be used in sentences which express *contingent* propositions, i.e. say things about the way the world happens to be. So the rules according to which they are used must make allowance for this connection with contingent fact, and this is a point that is missed by those who say that logical constants are governed by purely syntactical rules, that their use can be fully characterised by means of formal systems, and that logic can be reduced to syntax. Moreover, it can be argued that to speak of "truth", "proposition", "validity" etc., in connection with a formal system which in no way allows for the influence of contingent facts (how things happen to be in the world) on truth-values, is simply to generate confusion, since it obscures the fact that no such formal system could ever do what can be done by real languages, namely enable us to make statements *about* something non-linguistic.

23. Once we have seen that it is essential to propositional rogators that their applications do not *always* satisfy the NCD-condition, we are in a position to be struck, in a new way, by the fact that they sometimes do. How can their values sometimes be determined independently of contingent facts even though they are constructed or defined in such a way that contingent facts are to be relevant to their values? Or again, how is it that, starting with rogators whose values normally depend on contingent facts (e.g. "*p* or *q*", "not-*p*") we can construct new ones (e.g. "*p* or not-*p*") whose values never depend on contingent facts, whose applications always satisfy the NCD-condition? What I am getting at is that the necessary truth of a proposition can often be construed as illustrating the more general notion of satisfaction of the NCD-condition by the application of a rogator. And if we develop a theory of rogators, which describes and compares the different ways in which values of rogators may come to be determined independently of how things happen to be in the world (e.g. sometimes relations between the ways in which arguments are identified, sometimes relations between the method of identifying an argument and the rule for the rogator, sometimes only the way the rogator is constructed out of others, will be relevant), we may find (as I have found) that it is quite natural to say that there are different sorts of necessary truth, some of which can be described as "logical", some as "analytic", some as "synthetic". (This would provide an interpretation for a system of modal logic with different modal operators of different "strengths".) It is even to be hoped that studying the various ways in which the NCD-condition may come to be satisfied, and noticing their differences, may rid people of the inclination to oversimplify by saying that all necessity is due simply to language, or to conventions, or to syntax.

This may be illustrated by the following comparison. A rogator whose application to an argument identified in a certain way satisfies the NCD-condition is none the less a rogator, and the value which it takes is the very same thing as it may take in other applications not satisfying the NCD-condition. In particular, if it is a propositional rogator, and its application occurs in the construction of a proposition, then the mere fact that the NCD-condition is satisfied, e.g. if the proposition turns out to be one which is logically true, is no more justification for saying that what we have is not a proposition but a convention or rule, or for saying that it is not true in the same sense as other propositions, than there is for saying that the rogator is no longer a rogator, or that it does not have a value in the usual sense.

24. This completes my account of the applications of the notion of a rogator. I hope these rather condensed remarks show that we can look at some old problems in a new and illuminating way if we make the distinction between a function and a rogator.