Chapter Seven

KINDS OF NECESSARY TRUTH

Introduction

In chapter six I explained what is meant by saying that a proposition is analytic, and showed how it is possible to know that such a proposition is true independently any observation of facts. The features of an analytic proposition in virtue of which it is true ensure that it would be true in all possible states of affairs, so we can say that it could not possibly be false, that it must be true, that it is necessarily true, and so on. All these truth-guaranteeing features are topic-neutral and can be described in purely logical terms, such as that the proposition is made up of certain logical words in a certain order, with non-logical words whose meanings stand in certain identifying relations. This chapter will be concerned with the question whether there is any other way in which a proposition can be necessarily true.

In order to give this question a clear sense I must explain what is meant by "necessary", that is, give an account of the way in which the necessary-contingent distinction is to be applied. I shall start off by talking about the meaning of "possible". The next section will attempt to explain the meaning of "necessary". The rest of the chapter will be concerned to describe and distinguish kinds of necessary truths, and ways in which a proposition may be known to be true independently of observation of contingent facts.

(Throughout the chapter it must be remembered that this thesis is written from the point of view described in section 1.B.)

7.A. Possibility

7.A.1. We have reached the stage at which it is not enough to have only a rough intuitive grasp of the necessary-contingent distinction. If we are to make any further progress with the problem of synthetic necessary truth we must try to see clearly exactly what this distinction, or family of distinctions, comes to.

It is often pointed out that there is a close connection between the notion of necessity and the notion of possibility. A statement is necessarily true if it would be true in all possible states of affairs, or if it is not possible that it should be false. This is sometimes put by saying that necessity is definable in terms of possibility and negation. I do not think the connection is quite as simple as some logicians would have us believe (See 7.B.1, 7.B.10). There certainly is a close connection between the two notions, however, so I shall try, in this section, to explain how we can understand talk about possibility, or about "what might have been the case".

In order to do this, I shall make use of the very general facts which, in chapter two, I argued to be presupposed by statements about the meanings of words in English and similar languages. (See section 2.B, especially 2.B.6.) These are facts such as that our sentences describe states of affairs which can be thought of as made up of material objects possessing observable properties and standing in observable relations. More specifically, I shall rely on some of the arguments in

2.D to the effect that in this conceptual scheme universals (properties and relations) are not essentially tied to the particular objects which happen to instantiate them. (Cf. 2.D.5,ff., 3.C.3,ff.)

7.A.2. It is worth noting that the notions of necessity and possibility are not merely technical notions invented by philosophers, for we are all able to use the following words and expressions: "necessary", "necessarily", "possible", "impossible", "must", "had to happen", "couldn't have happened", "cause", "if so and so had happened", "if only I had done so and so ...", etc. Think of the words of the popular song: "That's the way it's got to be"!

Despite their familiarity, these notions are puzzling because they are "non-empirical" in a strange way. We can point to what is the case, but we cannot point to what isn't the case and might have been. Worse still, we cannot point to what is not the case and could not have been. At any rate, we cannot produce examples to be looked at, in the way in which we can produce or point to actual observable states of affairs, or events. How then do we learn to understand these kinds of expressions in the first place? The clue seems to be provided by a fact pointed out in 2.C.6, namely that in order to decide that something or other is possible, we have to consider properties, or, more generally, properties and relations and ask whether they are connected or not. (There are really many different kinds of necessity and possibility: I shall not discuss them all.)

7.A.3. Let us consider some examples. The piece of paper in front of me is not blue and square, but it might have been, or at any rate there might have been a blue and square piece of paper in front of me. The piece of paper which is in front of me is white and oblong, but it might have been different. There is a cardboard box on my table; it has a lid which is neither white nor oblong, though it might have been both. There is no paper on the floor near my chair, but there might have been, and it might have been either white and oblong or blue and square (or it might have had other shapes and colours).

What lies behind all this, is simply the fact, to which I have already drawn attention,[1] that universals are not essentially tied to those particular objects which happen to instantiate them. Universals are not extensional entities, they exist independently of the classes of objects which actually possess them. As remarked previously, one can have a property in mind, think about it, attend to it, recall it, associate a word with it, talk about it, etc., without thinking about any actual particular object which has that property. Neither the property of being blue and square nor the property of being white and oblong is essentially tied to the particular material objects which actually have them. Nor are they essentially tied to the times and places at which, as a matter of fact, they can be observed. When we see the properties of objects, or the relations in which they stand, we can see that they are not the sorts of things which have to occur where they do occur.

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1. (2.B.6, 2.D.5, 3.C.3, etc.)

7.A.4. This possibility of recurrence is, after all, what makes us describe properties and relations as <u>universals</u>, and contrast them with <u>particulars</u>. We can isolate out three aspects of their universality.

First of all there is actual recurrence. The whiteness of the piece of paper on which I am typing is a property which it shares with many other objects existing at the same time and at different times. Secondly, other objects exist which, although they are not instances of the property, might have been. The box on my table is not white, but it might have been. Thirdly, there might have existed objects which do not in fact exist (there might have been a piece of paper on the floor next to my chair), and if they had existed, then they might have had these properties. If there had been a piece of paper on the floor next to my chair, it might have been white.

Some philosophers would explain the universality of properties in terms of the first of these three aspects, namely actual recurrence, but this will not do, for there are probably properties, such as very complicated shapes, which are, as a matter of fact, instantiated by exactly one object, or possibly by no objects at all.

7.A.5. Universals <u>can</u> recur, even when they do not in fact do so. Now how do we know this?

When I look at an object and pay attention to one of its properties which does not have any other instances, how do I tell that that property is the sort of thing which could occur elsewhere, even though it does not. Is it simply a generalization from experience? Is it because I have seen many objects which share properties that I come to believe that this specific property at which I am looking is also the sort of thing which could be shared by several objects? If it were an empirical generalization, then I should have to leave open the possibility of an empirical refutation, or at least counter-evidence, but there does not seem to be any such possibility. Apart from the fact that I do not know what sort of experience would count as a refutation or as counter-evidence, the suggestion seems to be nonsensical because the sort of doubt which is appropriate to an empirical generalization does not seem to be appropriate here. (Indeed, the possibility of recurrence of universals is _presupposed_ by any empirical generalization.)

It seems that when I look at the shape or the colour of an object, I can _see_ then and there that what I am looking at is the sort of thing which can recur, since there is nothing about it which ties it essentially to this object, or this place and time.

7.A.6. When I look at a colour and see that it is the sort of thing which could occur in other objects at other places and times, I do this by abstracting from the _particular circumstances_ of its occurrence, such as the fact that it is possessed by this piece of paper here and now, is being looked at by this person, can be found to be two feet away from that particular table, and so on.

I believe that this sort of abstraction is often confused with another kind, namely abstraction from _specific features_, for example in Kant's remark (in C.P.R, A.713, B.741):

> "The single figure which we draw is empirical, and yet it serves to express the concept without impairing its universality ... for in this empirical intuition we consider only the act whereby we construct the concept, and abstract from the many determinations

> (for instance, the magnitude of the sides and of the angles), which are quite indifferent as not altering the concept 'triangle'."

I think Kant confuses two things which are very often confused, but which must be very carefully distinguished, namely universality and generality, both of which may be involved in one universal (or concept) "expressed" by a particular material object. Universality is common in the same way to all properties, but some properties are more general (or less specific) than others. The universality of a property consists in the possibility of its occurring in other objects than those which actually instantiate it. The generality of a property (or concept) consists in the possibility of its occurring in several different objects in different forms. I do not know whether there is a sense in which properties can be thought of as general or non-specific in any absolute way, but there is certainly a relative general-specific distinction. The property of being a triangle is more general, or less specific, than the property of being an isosceles triangle. A specific shade of red is more specific, or less general, than the hue, redness. (Cf. 3.A.1.)

In order to perceive the (relative) generality of a property we have to abstract from the specific features of an object which has that property. In Kant's example, we have to abstract from the specific ratios between the sides of a triangle and the specific sizes of the angles, its specific orientation, and so on, in order to perceive the generality of the property of being triangular. But this sort of abstraction is not what concerns us at present: we are interested not in abstraction from specific features, but in abstraction from particular circumstances: that is what occurs when we see a property as the sort of thing which can recur, whether it actually does so or not, that is, as a universal. It is presupposed by the other kind of abstraction, for only if it makes sense to talk of a property as being possessed by other objects does it make sense to talk of other objects as possessing other determinate (specific) forms of this property.

7.A.7. It should be clear that I am not trying to define the notion of possibility or "what might have been" in terms of what can be conceived or imagined: I am not saying "P is possible" means "P can be imagined". Imaginability is not a criterion for possibility nor vice versa. There may be things which are possible though no human being can imagine them, either owing to lack of experience, or owing to complexity. There may be colours which have never yet been seen and cannot be imagined at present, or shapes too complicated to be taken in. Worse still, people have imagined or conceived of things which later turned out to be impossible.

For example, one has to be a rather sophisticated mathematician to be unable to imagine trisecting an angle with ruler and compasses, and in the sense in which most of us can imagine that, it is surely possible for someone who is still more unsophisticated than we are to imagine seeing a round square. (Someone might draw a straight line and say that it was a picture of a round square seen from the edge!) So it will certainly not do merely to say that what is possible is what can be imagined, or that what is necessary is what could not be conceived to be otherwise: it will not do to offer this as a definition.

What I am saying is rather this: look at what goes on when you imagine what it would be like for something to be the case, and then you will see more clearly, from a philosophical point of view, what it is to describe a state of affairs as "possible". The important thing is that the various properties (and relations) which we can see in the world need not be arranged as they are, in the instances which they happen to have, and we acknowledge and make use of this fact when we imagine non-existent states of affairs, or when we talk about them or write stories about them or wish for them, or draw pictures of them. In short, that which makes imaginability possible in some cases, is what explains how states of affairs may be possible though not actual, namely, the loose connection between universals and their actual instances.

7.A.8. All this may be used to explain the notion of the set of "truth-conditions" of a proposition. We have seen (cf. 5.E.1) that in general whether the proposition expressed by a sentence S is true or not depends on three things:

(a) the non-logical words in S and their meanings,

(b) the logical form, and corresponding logical technique

(c) facts, or the way things happen to be in the world.

This shows that when a logical form, and a set of non-logical words are combined to form a sentence expressing a proposition, the linguistic roles of the logical constants, and the semantic correlations governing the non-logical words together determine a set of possible states of affairs in which to utter the sentence would be to make a true statement. These are the "conditions" in which applying the logical technique for determining its truth-value would yield the result "true". There is not usually just one truth-condition, as pointed out in 6.F.11. Every statement ignores some aspects of the states of affairs in which it would be true. Variations in these "irrelevant" aspects help to increase the size of the class of possible states of affairs in which such a statement would be true. Whether it is true or not depends on whether one of these possible states of affairs actually obtains, i.e. is a fact. (This can be generalized. If R is any rogator, and T is one of the values which it can take, and if A is an argument-set for that rogator, then R and A together determine a class of possible states of affairs, or T-conditions, namely those in which applying the technique for determining the value of R for A would yield the result T. "Rogator" was defined in 5.B.6.)

7.A.9. By taking note of the fact that universals can recur, that is by abstracting from the particular circumstances in which we see shapes, colours, and other properties, we are able to learn such things as that this book might have been the colour of that one, there might have been a box on the floor the same shape as the one on the table, there might have been pennies in my right-hand pocket instead of in my left-hand pocket. It should be stressed that there is nothing mysterious in this: apprehending the universality of a shape or colour or other property, does not involve making use of "inner-eyes" or other occult faculties: it is just a matter of using ordinary intelligence and ordinary eyes and imagination. We thereby take note of very general facts but for which our language, thought, and experience could not have been the sorts of things which they are. (See chapter two, section B.)

7.A.10. All this shows that there is a <u>non-linguistic</u> kind of possibility. By this I mean merely that when we talk about possibilities we are not talking about combinations of words which are permitted by the rules of some language. Contrast this with Schlick's remark (Feigl and Sellars, p.154): "I call a fact or process 'logically possible' if it can be <u>described</u>, i.e. if the sentence which is supposed to describe it obeys the rules of grammar we have stipulated for our language."

The class of possible states of affairs is much more complex and numerous than the class of sentences formulable in any language. Sentences are discrete and individually describable, and, at any one time, either finite in number or able to be arranged in a fairly simple sequence, unlike possible states of affairs, which shade continuously into one another in many different dimensions. (Austin: "Fact is richer than diction.")

7.A.11. Further, it should be noted that the concept of possibility cannot be reduced to that of logical possibility or analytic possibility. To say that a proposition is not a formal or analytic falsehood is to say that one cannot show it to be false merely by considering the meanings of words and the logical techniques of verification corresponding to its logical form. This simply means that observation of the facts may be relevant to determining its truth-value. It does not imply that any state of affairs is a possible one, or that there is a non-empty class of possible states of affairs corresponding to it as truth-conditions. For the question whether, for some other reason, the truth-value would come out as "false" in all possible states of affairs is still left open. At any rate, some argument is required to show that it is not left open: and that shows that there are different concepts of possibility here.

To sum up, knowing what possibility is, is not a matter of knowing the laws of logic and seeing which descriptions of possible states of affairs do not contradict them, neither is it a matter of knowing which combinations of words are permitted by the rules of some language. It is a matter of knowing that the world is made up of material things and their properties and relations, and knowing that these properties and relations are not essentially tied to those material things which actually instantiate them, that they need not occur in the arrangements in which they do occur, or at the places and times at which they do occur. Other factors might have been taken into account, such as loose ties between particulars and the actual places and times at which they exist. (This table is here now, but it might have been next door.) These factors will be ignored, not being relevant to our main problem.

(Note. This conception of possibility can be used to solve some philosophical problems. For example, it makes puzzles about the identity of indiscernibles disappear. If "indiscernible" means "could not possibly be different in some respect", then the principle that indiscernibles are identical is true. If "indiscernible" means "is not actually different in some respect", then the principle is false. A sphere in an otherwise empty universe will have two halves, despite its symmetry, because one of them could be a different colour from the other, even if it is not. If the principle were correct in its second sense, then, although an unsymmetrical object could exist alone in the universe, it could not be gradually transformed into a sphere, for on becoming a sphere it would consist of two (or indefinitely many - if we take account of sectors) parts which have all their properties and relations in common. What would happen to it then? It is clear that the principle is absurd.)

7.B. Necessity

7.B.1. The concept of possibility has been shown to have an application on account of the loose tie between universals and actual instances.

But understanding talk about possibilities is not enough for an understanding of talk about necessity. For that, one must know what is meant by "the range of all possibilities", or "what is not possible". The use of negation or the word "all" has to be defined afresh in these modal contexts, and corresponding to different ways in which its use is explained there may be different kinds of ranges of possibility, different kinds of necessity, different kinds of impossibility. We must therefore proceed with caution.

7.B.2. We have seen that the concept of "analytic" possibility is not very substantial (7.A.11). Similarly, the kind of necessary truth which we have found in analytic propositions does not seem to affect the range of all possible states of affairs, since the necessary truth of such a proposition is merely a matter of its having certain general features which ensure that it comes out as true, no matter what particular things or kinds of things it is about, and no matter how things happen to be in the world. So its necessary truth is not due to anything at all specific which has to be the case in all possible states of affairs. Hence its being necessarily true imposes no special restrictions on what may be the case: it does not seem to limit the range of all possible states of affairs in any way.

Let us try to find a more substantial and more general concept of necessity by following up what Kant said in the "Critique of Pure Reason" (B.4), namely that a statement is necessarily true if

> "it is thought with strict universality, that is in such a manner that no exception is allowed as possible, it is not derived from experience."

(The last clause, "not derived from experience", will be ignored for the time being. See Appendix VI.)

7.B.3. What is strict universality? How can no exception be allowed as possible?

Suppose the following were true: (1) "All triangles are red". It would then be a universal truth with no exceptions, but it would not be <u>strictly</u> universal, since it is clear that triangles do not have to be red: even if all triangles happen to be red, I can see, just by looking at a triangle that although it is not green it might have been, while still a triangle. Although there are no exceptions, nevertheless exceptions are allowed as <u>possible</u>. By contrast, the proposition (2) "All squares have exactly four angles" is not only a universal truth without exceptions, but it is strictly universal, since there could not be any exceptions: even if there were any other squares than the ones which there actually are, they would all have exactly four angles. (2) is a necessary truth, whereas (1) would not be necessary, even if it were true.

7.B.4. This can be expressed more generally if it is tied up with some of the remarks in 7.A about possibility and properties. First of all, let us consider propositions of the form "All P's are Q's", where "P" and "Q" are descriptive words referring to observable properties. Now recall that the universality of a property has three aspects (7.A.4). Firstly, the property may occur in several different actual particular objects. Secondly, it might have been instantiated in some of those particular objects which are not in fact instances. Finally, there might have been objects, which if they had existed, might have had the property. We may therefore say that the property referred to by "P" is <u>necessarily</u> connected with the property referred to by "Q", and the proposition "All P's are Q's" is <u>necessarily</u> true, if, and only if, all the objects in the first class, namely all those which actually have the property referred to by "P", also have the property referred to by "Q"; all the objects in the second class, that is all those which might have had the property P, would, if they had had it, also have had the property Q; and, finally, all the objects in the third class, namely those which might have existed though they do not, would, if they had existed and had the property P, also have had the property Q. In short, there are three sorts of potential counter-examples to a proposition of the form in question, namely those objects which have the property P, those which do not, but might have had it, and those which, if they had existed, might have had it; and to say that no exceptions are allowed as possible, is to say that none of these objects, if it had (existed and) had the property P would have been without the property Q.

7.B.4.a. We can generalize this further if we recall that only propositions are being discussed which are universal in form and mention no particular objects (see Appendix I). For a sentence made up only of logical constants and descriptive words referring to properties is true if and only if certain relations hold between the classes of objects possessing certain properties, relations such as inclusion, or mutual exclusion. Such a proposition is _necessarily_ true, then, if all the classes of objects with the specified properties do in fact stand in the specified relations, and, in addition, they would do so even if other objects had the properties in question than the ones which actually do so, even if there were other objects in existence than the ones which there actually are. In such a case not only are there no exceptions, but, in addition, no exceptions are possible.

This could be generalized a stage further to include propositions referring not only to properties, but also to relations, such as "two feet away from", "brighter than", "inside", and so on, but I shall leave that to the reader.

It should be noted how this definition differs from the definition of an analytic proposition: here we make no mention of "identifying relations" between meanings, nor do we restrict the sources of necessity to topic-neutral features of propositions. Thus, it is so far an open question whether all necessary truths are analytic.

7.B.5. All this may suffice for a _definition_ of "necessity", but it is not very helpful, since it does not _explain_ how it comes about that any statement is necessarily true, or how we can ever tell that it is.

What is missing is an explanation of how we can tell whether a counter-factual conditional statement asserting that no exceptions are allowed as possible is true, that is, how we can tell what would have been the case if certain objects had had certain properties. How do I tell that if the piece of paper on which I am now typing had been square then it would have had four angles? How do I tell that if there had been a tetralateral block of wood on my table then it would also have been tetrahedral? (Cf. 2.C.8.) It should not be assumed that simply because I know how to tell that something or other might have been the case, I know how to tell what else would then have been the case, or that I know how to tell what would be the case in all possible states of affairs. (See 7.B.1.)

There seems to be so complicated a range of possible worlds and possible states of affairs that it is hard to see how anything at all could be excluded from the range. There might be worlds in which space had five dimensions, or only two. There might have been a world in which there were only sounds, and no space or spatial objects or spatial properties (see Strawson's "Individuals", chapter two). There might have been worlds in which properties and relations existed which were quite unlike anything we can imagine. Or might there?

It seems clear that there is a tangled and complicated question here, which is not really relevant to such problems as concern us, for example, the problem whether it is both necessary and synthetic that two properties which actually do exist always occur together (such as the property of being four-sided and the property of having four angles). The source of the trouble is that there are different concepts of necessity, and different kinds of ranges of possibilities.

7.B.6. But our definition of "necessary truth" was restricted in such a way that we need not take account of all these complexities, for it is concerned only with classes of objects possessing properties which actually do exist in our world. We therefore have no need to talk about all possible worlds, since we can limit ourselves to talking about all possible states or configurations of <u>this</u> world, where "this world" describes a world in which the same observable properties and relations exist as exist in our world. (It should be recalled that the existence of universals need not involve actual existence of instances. See section 2.D.) Thus, since we are talking only about states of <u>this</u> world, we need not consider worlds without space and time, or five-dimensional worlds. (Compare what Kant says about his Copernican Revolution in the Preface to the second edition of C.P.R. B.xvi-xvii, etc.) (See note at end of this section.)

7.B.7. Now we may return to an explanation of how it is possible to tell that a statement is necessarily true. Once again we shall make use of general facts about universals, that is, observable properties and relations.

It has already been pointed out (7.A.4,ff) that a property exhibited by an object is the sort of thing which can recur. Now we must notice further that one object may possess more than one property at the same time. A material object may be both red and round. It may be cubical and transparent. It may be cigar-shaped, glossy and green.

When two or more properties are exhibited by an object, we may be able to see that some of them have no connection with the others. For example, the fact that a box is cube-shaped has nothing to do with the fact that it is red. Not only could the property of being cube-shaped occur in other objects, in addition it could occur in other objects without the colour which accompanies it in this one. Even if neither of these properties did occur without the other (which, of course, is not the case), we could still nevertheless see that there might have been an object which was cubical without being red, or red without being cubical. One need not have seen either of the properties actually exhibited without the other in order to see that they are capable of occurring separately, any more than one must have seen the shape or the colour in another object at another time or place in order to see that it can have other instances. (Cf. 7.A.5.) All we need is our eyes and intelligence, and the knowledge of what it is to be cube-shaped and of what it is to be red, and then one can see that it is possible to recognize either property in an object without its mattering whether the other is there or not.

Similarly, where there are two properties which we have never, as a matter of fact, seen in the same object at the same time, we may be able to tell that there could be an object with both of them. I have never, as far as I know, seen an object which is both cigar-shaped and blue, but there is nothing in either property, insofar as it is an observable property, which excludes the presence of the other. I know what it would be like to recognize both properties in one and the same object.

Thus, even if the two statements "All cubes are red" and "Nothing cigar-shaped is blue" are true, that is have no exceptions, nevertheless they do allow exceptions as possible. We can see, by examining the properties concerned, that they are not necessarily connected. It is by contrast with this sort of case that I shall explain how statements can be necessarily true.

7.B.8. We have added a refinement to our concept of possibility by taking note of the fact that not only are universals not essentially tied to their actual instances, so that they can be instantiated in other places and times than they are in fact, but, in addition, they are not essentially tied to one another, so that they can occur in different combinations from those in which they in fact occur. Universals are unfettered by their instances, and also, sometimes, by one another. Not always, however, and limitations on this second sort of freedom generate the kind of necessity which is of interest to us. There may be something in the constitution of a property which ties it to another property, or which prevents its occurring with another property. If so, this may have the consequence that a statement using words which refer to those properties is not only true, but necessarily true, since no exceptions are allowed as possible. (Exceptions would be objects in which these tied properties occurred separately, or in which incompatible properties occurred together.)

If there are any such relations between properties which are not identifying relations, then they will provide us with a new class of relations between descriptive words referring to properties so related, and here, as in the case of analytic propositions, the relations between descriptive words, together with features of the logical technique for discovering truth-values of a statement, may determine the outcome of applying the technique in any possible state of affairs (cf. 6.C.1.). So if there is some way of knowing that the properties referred to by words stand in such relations, then we may be able to determine the outcome of an empirical investigation to discover the truth-value of a proposition, without actually making that investigation. If this is so, then we shall have found a new type of illustration of the fact pointed out in 5.E.1 and 5.E.2, that although in general the value of a rogator depends on (a) the arguments, (b) the technique for discovering values and (c) the facts, nevertheless there are cases where without knowing any facts (i.e. without having any empirical knowledge of how things happen to be in the world) we can discover the value by taking note of relations between the arguments and examining the general technique for determining values. We shall have found a way of telling, without knowing which particular objects there are in the world, nor what properties and relations they instantiate, that none of them is an exception to what is asserted by some statement. That is, we shall have found a new kind of <u>a priori</u> knowledge of the truth of a statement. (See end of 7.B.2.)

7.B.9. If there are these connections between properties, and if we can know that they exist, for example by <u>examining</u> the properties in question, then this will explain how we can be in a position to assert such statements as "If this had been square then it would have had four angles", or "If this had been turquoise, then it would not have been scarlet". Thus, by talking about properties, and their ties with or independence of one another and their instances, we are able to explain some uses of the words "necessary" and "possible", and counter-factual conditional statements.

To sum up: since properties are not tied to their actual instances we can talk about what might have been the case in the world, and since they may be tied to one another (this includes incompatibility), we can talk about what would have been the case if so and so had been the case. Hence we can talk about statements to which no exceptions are possible, that is, statements which are necessarily true.

7.B.10. All this should show that the concept of necessity is far more complicated than the concept of possibility. (See 7.A.1, 7.B.1.) Only the latter is required if we are to be able to use our ordinary language to describe new situations, to ask questions about unknown facts, to understand false statements. We need only understand that the range of things which might have been the case is wider than the range of things which are the case. The concept of necessity is required when we grow more sophisticated, when we wish to do more than simply describe what we see or ask questions about what is to be seen in the world. It comes in when we wish to draw inferences, when we wish to know about the properties of all things of a certain kind without examining them all, when we do mathematics or philosophy, or try to explain what "makes" things happen as they do, or when we ask whether happenings are avoidable or not. It comes in also when we try to justify the assertion of a counterfactual conditional statement, about what would have happened if something or other had been the case. In order to understand talk about possibility, one need only see that states of affairs are possible which are not actual, whereas in order to understand talk about necessity, one must, in addition to understanding talk about possibility, also see the reasons why the range of possibilities is limited in certain ways. The former requires only a perception of the loose tie between all universals and their actual instances (by abstraction from particular circumstances), the latter requires perception of the strong ties between some universals and other universals.

(7.B.note. It should not be forgotten that in all this we are talking only about the kind of necessity which arises out of limitations on the possible states of <u>this</u> world, in which objects have properties and stand in relations only of the sorts which are capable of having instances in our world. There may, however, be other kinds of necessity, other kinds of limitations on what may be the case in the observable world. (Compare 7.B.6.)

For example, there may be limitations on the range of possible states of affairs - or possible worlds - which can be talked about in a language using a distinction between subjects and predicates. Or there may be limitations on what can be the case in states of affairs which are observable by beings with senses of any kind. (E.g. a sense which enables them to perceive magnetic and electrical properties directly.) Perhaps there is some other kind of necessity, called "natural necessity" by Kant, which is operative when types of events or states of affairs stand in <u>causal</u> connections.

Kant talked also about a kind of necessity which involves <u>particular</u> objects, such as the necessity in the synthesis of an experience of a particular object (corresponding to the "form" of the object), but this sort of thing need not concern us. We have decided to ignore statements mentioning particulars - see Appendix I - and in any case the ascription of necessity to such statements can usually be explained in terms of their being instances of some universal statement which is necessarily true, as when we say "Tom's bachelor uncle must be unmarried".

There is no space here to discuss a sufficiently wide concept of necessity to allow us to take account of all these cases and such questions as whether it is necessary that space is three-dimensional. It is not clear to me that there is a perfectly general and absolute concept of necessity. For example: if a statement is necessarily true, then it is not obvious that it makes sense to ask whether it is necessarily necessarily true. See end of Appendix I. There may be only a relative concept, operating at different levels, each level being characterized by the type of thing which can count as the reason why a statement is necessarily true. At the level which concerns us, the reason must be that there is a perceptible connection between observable properties or relations.)

7.C. Synthetic necessary connections

7.C.1. This chapter has so far shown, by drawing attention to certain features of the conceptual framework which we presuppose in using descriptive words and referring expressions (cf. 2.B.4-6), how we can understand talk about possibility, and, in a vague way, what is meant by saying that some statements are necessarily true. A statement of the sort under discussion (using only descriptive words and logical constants) is necessarily true if there are connections between the properties referred to by the descriptive words, which ensure that no particular object could be a counter-example to the statement, since certain combinations of those properties in one object are ruled out by the connections between them. Now we must ask whether all such connections between properties are identifying connections (see 6.C) or whether some non-identifying or synthetic connections between properties can ensure the necessary truth of a statement.

7.C.2. Let us be clear about what we must look for. If knowing the meanings of words (sharply identified meanings, that is, see 6.D.3 and section 2.C), suffices, on its own or together with purely logical (topic-neutral) considerations, to justify the claim to know that properties are related in some way, then that is an identifying relation, not a synthetic relation. For relations between properties to be synthetic, knowledge of them must require something more than the knowledge of which properties they are, and the "something more" must not be purely logical. But what more could there be? Is there some way of examining properties themselves (a non-logical enquiry, since it presupposes actual acquaintance with a special kind of subject-matter) in order to discover that there is a connection between them, a connection which need not be known in order to know which properties they are? We must now investigate some examples, and see whether this sort of insight is possible. If any such insight is possible, it will explain Kant's talk about "appeals to intuition". (See 6.B.2, 6.C.11.)

7.C.3. The most interesting examples come from geometry, though there are many other kinds which cannot be described here for reasons of space. (More examples will be found in Appendix V. See also 2.C.8, 3.C.10 and 3.D.10).

In 2.C.8. we defined the words "tetralateral" and "tetrahedral", the former referring to the geometrical property of being bounded by four plane faces, the latter to the property of being bounded by plane faces and having four vertices. I argued that the two words refer to two different properties which can be identified independently of each other, since one can notice either property, attend to it, think about it, or talk about it, without being aware of the existence of the other. So in order to know that they are inseparable it is not enough to know which properties they are: in addition one must carry out some sort of construction, either in imagination or with sheets of cardboard, or with diagrams, or somehow examine the two properties, in order to be sure that all possible ways of putting four plane surfaces together to bound a closed space must result in there being exactly four vertices, and that no other number of plane surfaces can yield exactly four vertices. This examination presupposes acquaintance with a special kind of property, and cannot take a topic-neutral form. It does not, therefore, involve drawing conclusions in a purely logical way, so cannot account for knowledge of an analytic truth, according to the definition of 6.C.10.

I call such a construction, carried out for the purpose of enabling oneself or someone else to perceive the connection between two or more properties (or relations), an "informal proof". (For more detailed remarks see next section.)

7.C.4. It seems, therefore, that since an informal proof of a non-logical kind is required, in addition to a specification of the meanings of "tetrahedral" or "tetralateral", for a justification of the assertion (1) "All tetralateral objects are tetrahedral", this must be a synthetic statement. Its justification is quite different from the justification of (2) "All gleen things are glossy", which proceeds by specifying that the word "gleen" refers to a combination of the property referred to by "glossy" with another property (that referred to by "green", say), and then taking account of purely logical properties of the technique for verifying statements of the form "All P things are Q". There is no identifying relation between the meanings of "tetralateral" and "tetrahedral", from which a logical proof of (1) could proceed.

7.C.5. There are many more examples of this sort of connection between properties, some more problematic than others. Here are a few. (In most cases "improper" or synthesized properties are involved.)

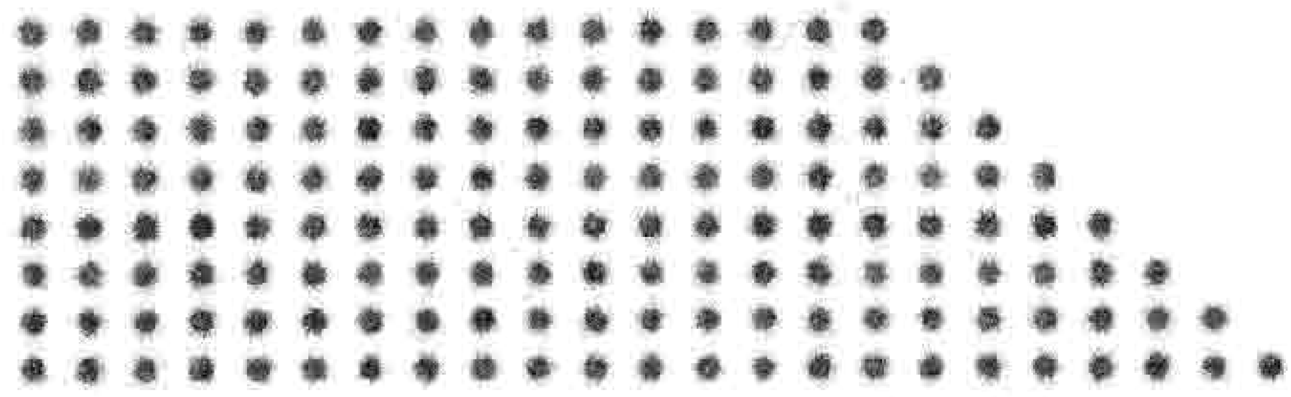
(a) The property of being bounded by three straight lines is necessarily connected with the property of being a plane figure with three vertices.

(b) In 3.D.3 and 3.D.4 we described two different "procedures" for picking out triangular shapes, of which the first involved memorizing one triangular shape and picking out others on account of their deformability into it, while the second involved pointing at sides in turn and reciting "Bing bang bong". Here are two synthesized properties which seem to be necessarily connected. Can the connection be shown, by purely logical considerations, to follow from identifying relations?

(c) No closed space is bounded by three planes. Is the incompatibility between the property of being a closed space and the property of being bounded by three plane surfaces analytic?

(d) If a cube is inside a sphere and a piece of wire is inside the cube, then the wire is inside the sphere. Is the transitivity of the relation "inside" due to some identifying fact, or can one know which relation it is without being aware of its transitivity, or anything which logically entails its transitivity?

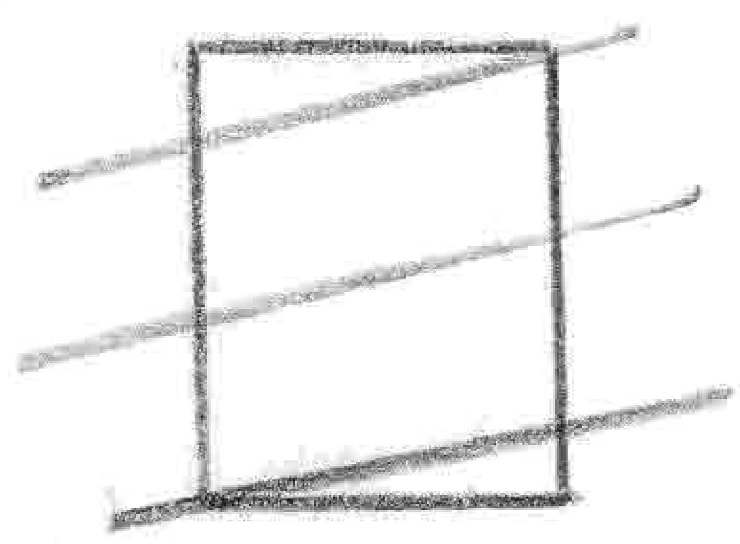
(e) Any pattern made up of regularly spaced rows of regularly spaced dots is also made up of a sequence of regularly spaced columns of dots.



It also consists of an array of diagonal rows of dots. (Diagonal rows inclined at various angles may be seen in the array.) All these several aspects of one pattern seem to be necessarily connected: the presence of some of them can be seen to entail the presence of others. Are these identifying connections between the aspects? Are they purely logical consequences of identifying connections?

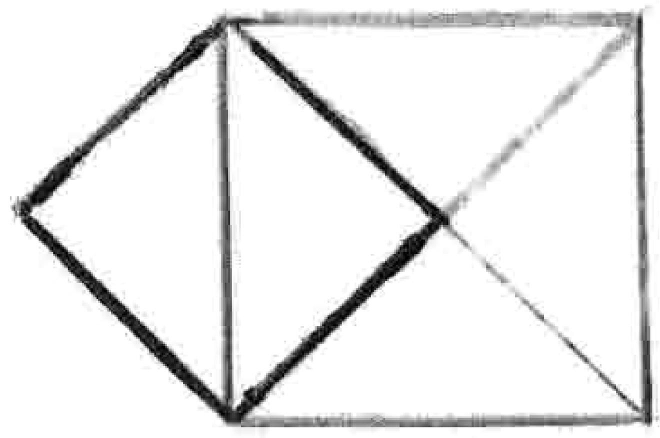
(f) Consider Kant's example; no left-handed helix may be superimposed on a right-handed helix.

(g) Consider Wittgenstein's example (R.F.M., Part I, 50.): any rectangle can be divided into two parallelograms and two triangles (by a pair of parallel straight lines passing through opposite corners, and a third parallel line between them). Is this due to some identifying fact about the meaning of "rectangle"?



(h) Any object with the property of having a shape which occupies the space common to three cylinders equal in diameter whose axes pass through one point at right angles to one another, has also the property of being bounded by twelve equal foursided faces, each of which is part of a cylindrical surface. (This is the shape obtained by pushing a hollow cylindrical cutter through a potato three times in mutually perpendicular directions in a symmetrical way.)

(i) If the side of one square is the diagonal of another, then the former can be divided into pieces which, on being rearranged, form two squares congruent with the latter.



In all these cases try, seriously, to say what the linguistic conventions are on which we must be relying implicitly when we perceive these necessary connections. (See Wittgenstein's "Remarks on the Foundations of Mathematics" for a serious attempt to meet this challenge. Cf. 7.D.10,ff., below.)

7.C.6. To all this the following objection may be made: "Of course there is no simple identifying relation between the meanings of the words 'tetralateral' and 'tetrahedral', and between the other words used in your examples, but this does not mean that the necessary connection between them is not a logical consequence of identifying relations between meanings. For the meanings of the words must be explained in terms of more fundamental geometrical words, such as 'plane', 'line', 'point', 'intersection', etc., and the meanings of these words stand in identifying relations, from which the connections to which you have drawn attention can be deduced by purely logical considerations." The objector will thereupon produce some axiomatic system of geometry, in which his "more fundamental" words occur as primitive or undefined terms, which he will use to define the words which interest us, and then, triumphantly, he will deduce from the axioms of his system, together with his definitions, using only logically valid inferences, that such statements as "All tetrahedral objects are tetralateral" are theorems.

But this is not enough. He must show first of all that his definitions do not simply take as identifying relations, relations which can be regarded as synthetic necessary connections. That is to say, it is not enough for him to show that words like "tetrahedral" can be defined as he suggests, he must also show that they have to be so defined, that it is impossible to understand them in some other way (e.g. by associating them with immediately recognizable properties), otherwise he will be arguing in the manner criticized in 2.C.10 and 3.C.10 (i.e. trading on ambiguities, and using loose criteria for identity of meanings.) In short, he must show that his definitions are definitions of the words they purport to define. Secondly, he must show that his "axioms" are in some sense true by definition, that they state or are logical consequences of identifying facts about the meanings of the primitive terms, and that they are not themselves statements which are necessarily true and synthetic. We could, of course, adopt additional verbal rules of the sort described in section 4.C to make the sentences expressing his axioms into expressions of analytic propositions, but he must show that only if such rules are adopted can the words in these sentences be understood as referring to those geometrical features to which they do refer. Once again: it is not enough for the objector to show that words can be defined in such a way as to make certain sentences express analytic propositions. He must show that as ordinarily understood they have to be so defined, or at least that unless they are so defined they cannot refer to properties which are necessarily connected.

7.C.7. It is far from obvious that there must be identifying relations of the sorts which could correspond to the axioms of an axiomatic system of geometry. (See remarks about superfluous "links between descriptive expressions", in 2.D.3 and 2.D.4.)

After all, we are not concerned with an abstract system of lifeless symbols having no empirical use, but with words which describe properties or aspects of physical objects which we can perceive and learn to recognize. (See 3.A.2) These words occur in ordinary everyday statements, such as "Here's a table with a square top", or "These three edges of this block of wood meet in a point", expressing contingent propositions which may be true or false. But (as pointed out in 2.D.2 note, and again in section 5.A), no system of axioms can suffice to give words meanings of this sort, for, in addition, semantic rules are required, correlating the words with non-linguistic entities such as observable properties. If we must have such semantic correlations, is it not conceivable that they may, on their own, suffice to give words their meanings, and determine their use and their relations to one another, without the aid of any further "axioms" or "linguistic conventions"? If so, it is surely an open question, requiring further investigation, whether all such relations are either identifying relations between meanings or logical consequences thereof. As remarked in 3.A.4. (cf. 3.D.9), in order to be able to use a descriptive word one must be able to recognize <u>some</u> universal immediately, so it is an open question whether some of these immediately recognized properties or features stand in relations with others, of a kind which must be discovered by examining them: why should the only things we can see using our eyes and intelligence be facts about particulars?

7.C.7 (note). It is very common nowadays to think that any geometrical proof must start from axioms which are all arbitrarily selected, serving as expressions of linguistic conventions of some kind, specifying the meanings of the geometrical words involved in them. The reason why people think this is that different systems of axioms may all be internally consistent, as is usually pointed out in connection with systems which do not include Euclid's parallel axiom. Consider Hempel's assertion (in Feigl and Sellars, p.243):

> "The fact that these different types of geometry have been developed in modern mathematics shows clearly that mathematics cannot be said to assert the truth of any particular set of geometrical postulates; all that pure mathematics is interested in, and all that it can establish, is the deductive consequences ....." (Compare Russell's definition of pure mathematics at the beginning of "The Principles of Mathematics." 2nd. Ed, p.3.)

All this, however, presupposes that internal consistency and deductive consequences are all that interest us, but it need not be all, if geometrical theorems are intended to state facts about observable geometrical properties. In that case, the axioms include words which refer to non-linguistic entities, and may be true or false, as well as consistent or inconsistent with one another. They are then not definitions, since the words in them are given their meanings independently, by correlations with different properties. It is not an accident that the kind of geometrical proof which involves not logical deductions from axioms, but the construction of diagrams, with construction-lines and sides and angles marked as equal, etc., occurs in a branch of what mathematicians call "pure" mathematics. Some philosophers (unlike Frege and Kant) give the impression that they are quite unaware of this, as is shown by the quotation from Hempel and many remarks which I have heard in discussions.

Perhaps it is wrong to think that by examining geometrical concepts or properties we can "discover" that the parallel axiom is true, but that is a very special case, since it concerns infinitely long lines (and therefore not ordinary observable properties), and it does not follow that other axioms are also merely matters of convention. For example, the "theorem" that every rectangle is divisible into two triangles and two parallelograms is on quite a different footing from the assertion that two parallel lines never intersect. It is not a mere defining postulate, and neither is it a contingent fact. (See 7.E for more on this.)

7.C.8. The argument so far may be summarized thus: at some point in the explanation of the meanings of descriptive words we must point to objects of experience with which they are correlated (i.e. we have to appeal to what is "given in intuition"). But then it is an open question whether these non-linguistic entities (properties) are so related as to ensure that some of the statements using words which refer to them are necessarily true, or whether all such relations must be identifying relations between meanings. This is not a question which can be settled merely by pointing to a set of axioms or linguistic conventions which could set up identifying relations and make statements analytic, for to say that they can do the job of making statements necessarily true is not to say that they are indispensible.

And to say that anything else which does the job must give the words the same meanings anyway, is to base a question-begging argument on the use of loose criteria for identity of meanings. (2.C.10, 3.C.10.)

I am trying to show that some very superficial and slipshod thinking lies behind many denials of the existence of synthetic necessary truths.

7.C.8. note. We are not interested in the question whether some statement in some actual language is or is not analytic, or whether the relation between certain sets of words in some actual language is an identifying relation. (Cf. 6.E.9.) This sort of question is of little philosophical interest and has to be based on an empirical enquiry in order to discover exactly what people mean by the words they use, and the discussion of chapter four and section 6.D shows clearly that there may be no definite answer to such a question, or there may be answers which can be summarized only in a statistical form. (Cf. 4.B.7.) We are concerned only with the question whether certain sorts of statements <u>have</u> to be analytic, or whether it is possible to give their words meanings which are identified independently of one another, and then discover, by examining the properties referred to, that the outcome of applying the logical techniques for determining the truth-values of such statements can yield only the result "true". Even if statements referring to such properties are analytic in English, or in some axiomatic geometrical system, owing to the fact that auxiliary rules have been adopted, setting up identifying relations between meanings, that does not prove anything, for the rules may be superfluous.

Failure to appreciate this point can cause people to argue at cross-purposes, for example over the question whether it is analytic that nothing can be red and green all over at the same time.

7.C.9. All this should at least show that the question whether some necessary truths are synthetic, on account of being true in virtue of relations between universals which are neither identifying relations nor logical consequences of identifying relations, is an open question. It has to be settled by a closer investigation of what goes on when one examines a pair of properties, such as the property of being tetralateral (bounded by four plane surfaces) and the property of being tetrahedral (having four vertices), and discovers, possibly with the aid of an informal proof, that there is some unbreakable connection between those properties. That is, we must look at what goes on when a person discovers, perhaps with the aid of an informal proof, that, owing to the relation between some properties (relations such as entailment or incompatibility), a sentence expresses a universal proposition which not only has no exceptions in fact, but which allows no exceptions as possible. (See 7.B.2, 7.B.4.ff.)

7.D. Informal proofs

7.D.1. So far, I have tried to show that just as we can see (using our eyes and ordinary intelligence, cf. 7.B.7) that the redness of a round and red object is able to occur elsewhere without the roundness, even if as a matter of fact it does not, so can we see that some properties are <u>unable</u> to free themselves from certain others, with which they are always found, or unable to cohabit with some with which they are never found. For example, I have argued that by examining the appropriate properties and discovering their relations we can detect the necessary truth of such statements as "All tetrahedral objects are tetralateral" or "No closed spaces are bounded by three plane surfaces." This shows that not only are we able to see that the actual state of affairs in this world[1] is not the only possible one, by seeing that universals are not essentially tied to actual particular instances nor to one another, but, in addition, we can see that there are some limitations on the ways in which these universals can be instantiated, some limits to what may be found in a possible state of the world. (This can be used to explain Kant's distinction between "form" and "content", in some contexts, and also, since it is concerned only with connections between properties and relations "tangible" to the senses, why he talked about "the form of <u>sensible</u> intuition". See <u>C.P.R.</u> A.20, B.34,ff; A.45, B.62; B.457 n.)

Section 7.C was concerned to establish that the necessary truths discovered in this way are not analytic, since first of all their necessity is due to non-identifying relations between properties, and, secondly, in order to become aware of them, one requires some kind of insight which is not purely logical, since it presupposes acquaintance with a specific kind of subject matter and is therefore not topic-neutral. For the first step I relied on arguments very like those of

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1. (7.B.6.)

3.C.9 and 3.C.5-7, to show that the properties are independently identifiable, and for the second I relied on the fact that the insight into the connections between properties always requires some kind of examination of those properties. In this section I wish to say a little more about what goes on when one examines properties, by talking about informal proofs. (7.C.3.)

7.D.2. What happens when I construct an informal proof to enable someone (possibly myself) to see that properties are related in some way? There is a very great variety of cases. For example:

(a) I might enable someone to see that nothing can be both circular and square by drawing a circle on transparent paper and getting him to try to draw a square on which it can be superimposed, in the hope that he will perceive the incompatibility of the two properties. I might point to a curved bit of the circle and a straight bit of a square and say: "This sort of thing can never fit onto that sort of thing".

(b) To show someone that if a triangle has two equal sides then it has two equal angles, I may point out that if it is picked up, turned over, and laid down in its former position with the sides interchanged, it must exactly fit the position it occupied previously since each of the two equal sides lies where the other was, and the angle between them does not change by being reversed. Hence each of the two angles which have changed places fits exactly on the position occupied formerly by the other, so the angles are equal.

(c) To show someone that if a figure is bounded by four plane surfaces then it must have four vertices, I may hold up sheets of cardboard and show that the only way to get four of them to enclose a space is first to form an angle with two of them, then to form a "corner" or pyramid without base, by adding a third, then to complete the pyramid by adding the fourth. He can then count the number of vertices, or corners. (This also helps to show why three planes cannot enclose a space.)

(d) To enable a person to see that if anything has both the property of being a "kite" (four-sided figure with a diagonal axis of symmetry) and the property of being a rectangle, then its shape is square, I may draw a kite and show that a pair of adjacent sides must be equal if it is symmetrical about a diagonal, and remind him that a rectangle with a pair of adjacent sides equal must be square.

(e) To enable a person to see that any rectangle has the property of being divisible into two triangles and two parallelograms, I may simply draw a rectangle, and then draw three parallel lines obliquely across it so that each of the two outer ones passes through one of a pair of opposite corners. (See 7.C.5, example (g).)

Owing to the enormous variety of cases, I shall be able to make only a very few rather vague and general remarks. (See Wittgenstein's "Remarks on the Foundations of Mathematics", for a detailed discussion of many more examples.)

7.D.3. First of all, I claim that each of these proofs is perfectly rigorous, and having once seen and understood it I am perfectly justified in asserting and believing the general statement which it is supposed to prove, such as "Nothing is both circular and square", or "Every rectangular figure is divisible into two triangles and two parallelograms". (The claim that such a proof is valid is a mathematical claim, not a philosophical one, since it is to be tested mathematically by trying to construct counter-examples: more on this presently.)

What happens when I see such a proof as a proof, when I see, as a result of going through the proof, that two properties are connected, and a universal statement necessarily true? The answer seems to be that I pay attention to a property, and notice that although it can be abstracted from the particular circumstances in which it is instantiated (see 7.A.6), so that it could occur elsewhere, and be recognized, even if it does not actually do so, nevertheless, it cannot be abstracted from the fact that it occurs in an object which has some other property (or from which some other property is absent). In particular, the construction of the proof may show me how, once I have found any other object which has the first property, I can repeat the method of construction of this proof in order to demonstrate that the other object has (or has not, as the case may be) the other property. The proof gives me a general principle for going from one property or aspect of an object to another, thereby showing me the reason why no exceptions to the proved general statement are "allowed as possible" (7.B.2,ff. 7.B.ff.) In Wittgenstein's terminology: the proof serves as the "picture of an experiment" (see R.F.M. I.36.) It may be better to say: the proof serves as a picture of a proof.

7.D.4. Perhaps we can see more clearly what goes on by distinguishing token-proofs from type-proofs. A token-proof is the particular event or set of marks on paper etc., spatio-temporally located, observed by you or me. The type-proof is a new universal, a property common to all token-proofs which use the same method of proof. The function of the token-proof is to exhibit the common property, the type-proof (a pattern); and to have grasped the proof, to have seen "how it goes", is to have seen this new universal. Now, the essential thing about the new property is that it is made up of various parts connected together (compare: the shape of a cube is a property made up of various parts connected together, such as the several faces - see section 3.D on non-logical synthesis). We may think of it as a "bridge-property" which connects one or more of its parts with others. Thus, when I start with a rectangle, draw three parallel lines, and end with a figure divided up into two parallelograms and two triangles, I have exhibited a bridge-property which starts from the property of being rectangular and goes to the property of being divided up in a certain way. This bridge-property is a <u>temporal</u> property, like the <u>tune</u> common to two occurrences of sequences of sounds (cf. 3.A.5.): it has to be exhibited by an "enduring particular". What the token-proof shows me, when it shows me that the property P is connected with the property Q, is that any object with the property P is capable of being used in a token-proof of the same type, since P is the starting point of a bridge-property which leads to Q. Thus the proof makes evident the connection between two properties by exhibiting them as parts of a new property. The token-proof shows how P and Q both "fit" into the type-proof.

This reveals a relation between words which are semantically correlated with those properties. In virtue of this relation some propositions using those words are necessarily true (Cf. 5.E.2, 6.C.1.).

7.D.5. But how do we discover the connection between the initial property and the bridge-property, or the connection between the bridge-property and the final property? How do we see that any object with the property P must be capable of being used in a proof of this type?

This is a crucial question. Consider a particular instance: how does the proof that every rectangle is capable of being divided into two triangles and two parallelograms show me that every rectangle is capable of being the starting point of such a proof? Do we need another bridge-property here, starting from the property P and ending in the former bridge-property? Obviously not. Then why not? The answer seems to be simply that a proof must start somewhere, and wherever it starts there must be something which is taken as not needing proof, namely that the first steps of the proof are possible. The reason why this needs no proof is that it may be discovered simply by inspecting the original property. Just by inspecting the property of being a rectangle I can see that if anything has the property then a line traversing it obliquely may be drawn through one of its corners. A person who cannot see even this will not follow the proof in question.

In other words, the account of the function of a proof in terms of type-proofs, or bridge-properties, is incomplete, since it leaves out the essential fact that at every stage of the proof something just has to be <u>seen</u> by examining a universal, namely that the next stage may proceed from there: this must be something which requires no proof. The whole point of a proof is to bring out a connection which is not evident. Where a connection is evident no proof is required, and this kind of connection which displays itself must be found at every "step" in a successful proof.

7.D.6. By pointing to a particular object I may draw someone's attention to some property or other universal instantiated in that object, but I cannot <u>force</u> him to see it. Similarly, by drawing his attention to a property or pair of properties I may succeed in drawing his attention to a connection between those properties in virtue of which one cannot occur without the other, or in virtue of which they are incompatible, but I cannot *force* him to see it. In some cases I can *help* him to see it by constructing a proof, by drawing his attention to a new property, a bridge-property of which the other two are somehow parts or constituents. But I cannot force him to see the bridge-property (I cannot force him to see what the type-proof is so that he could recognise it again in another instance: I cannot force him to see how the proof goes), and I cannot force him to see how it reveals a connection between the two properties in virtue of their connection with it.

This is very vague, and ignores differences between different kinds of proof. Perhaps it will be made a little clearer by the replies to some objections.

7.D.7. The first objection is that all my talk about "seeing" properties and connections between properties is far too psychological to serve as an account of what a proof is.

It is important to be clear about the sense in which the account is psychological and the sense in which it is not. Certainly the fact that someone takes a proof as valid is not what makes it valid (cf. 7.A.7, where a similar objection was raised to my account of possibility). But no account of what a proof is can avoid using psychological concepts such as "belief", "certainty", "understanding", "conviction"; for what is a proof supposed to do? It certainly cannot *make* the proposition true which it is supposed to prove. It cannot make it necessarily true either. The necessary truth of "Nothing is both round and square" in no way depends on the fact that anyone has ever proved it. Perhaps its necessary truth might be said to depend on the possibility of constructing a proof. But what makes this a possibility is a connection, or set of connections, between properties - which is precisely what is shown by the proof. The existence of the proof (token-proof) does not bring the connections between properties into existence, for the proof depends upon them for its own existence, and they are capable of ensuring the necessity of the proved proposition without the intermediary of the proof.

The proof neither makes the proposition true, nor makes it necessarily true. Rather, it brings out the reason why it is necessarily true. "Bringing out" can only mean "making evident to someone or other", for the reason is there, doing its job faithfully, whether any proof is constructed or not. So the role of proof is to enable someone to see that a statement is true, or necessarily true, and it follows that psychological concepts must be employed in a description of what proof does and how it does it.

7.D.8. The error in the objection is to assume that because psychological terms are used to explain what a proof is, the validity of proofs is a psychological matter. But this is not so. (Necessity is not defined in terms of inconceivability, but both are explained together. See 7.A.7.)

A person cannot simply turn up and say that he knows that it is possible for a round square to exist because he has seen and been convinced by a proof of its possibility. If he has seen a proof, and followed it, then he has become acquainted with a new universal (the type-proof: 7.D.4), and, as pointed out in 7.A, a universal is the sort of thing which can recur, so he must be able to reproduce the proof and point out its relevant features to us. He cannot get by with the remark that he remembers how the proof goes, and can imagine it, but cannot produce it for us, for what can be imagined by him proves nothing unless it is the sort of thing which could be drawn on paper, or otherwise concretely reconstructed and subjected to scrutiny. Neither is it enough for him to draw a straight line and say that it is a picture of a round square seen from the end, for he must explain in virtue of what this can serve as a picture of a round square, i.e. how it exhibits the roundness and the squareness of the thing it is meant to represent.

7.D.8.a. Of course, a person may produce a perfectly valid proof which, for the time being, no one else can follow, on account of its complexity. But this does not make its validity a subjective matter, any more than the possession by an object of a complex property (e.g. a shape, or other structure) is a subjective matter in cases where only one person happens to be able to "take in" the property. We all know, at least in a vague way, what would count as an objective refutation of the proposition alleged to be necessarily true, for if we understand the proposition then we are able to recognize counter-examples, should they turn up. (This must be modified to take account of existence-proofs, or proofs of possibility.)

Even if the proposition happens to be necessarily true, but not validly proven, we know what would count as a public demonstration of the invalidity of the proof, for there must be something _about_ the proof in virtue of which it is supposed to establish the connection in question (i.e. there must be type of which it is a token), and it could be shown to be invalid by a construction based on the same principles (a token of the same type) which leads to a proposition which has demonstrable counter-examples.

(A full account would describe and classify various ways in which one may fail to see the validity of a valid proof, or come to think an invalid proof is valid. E.g. one may think one has seen a property which one has not seen. One may have seen _a_ bridge-property, but not one which does quite what it is taken to do, as when one notices a connection which works in most cases without seeing that there is a special class of counter-examples.)

7.D.9. Next it may be objected that what I say is just wrong, since what really goes on in a proof is that, in a "dim" way, we are shown how a _formalized_ proof would go, starting from identifying relations between concepts and drawing purely logical conclusions, without any need for such things as "showing" the connections between properties. Since it looks as if I am not doing any such thing when I use an informal proof to enable someone to see that all tetralateral objects are tetrahedral, it is up to the objector to say _what_ the formalized proof is that I am unwittingly presenting, e.g. by showing me the identifying relations or definitions from which he thinks I am drawing purely logical conclusions, whereupon the arguments of 7.C.6 will come into play.

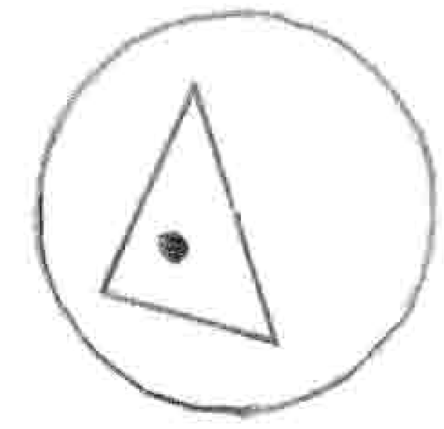
Secondly, the objector will find himself in difficulties as soon as we ask how the conclusion <u>follows</u> from those statements of identifying facts which he claims to be implicit premises in the proof. For, as pointed out in 5.C.9 and 5.C.10, etc., in order to see that some inference is logically valid, it is necessary to perceive properties of general logical techniques corresponding to the logical forms of propositions, or connections between such techniques, and this is just the sort of insight into the connections between universals which is provided by the informal proofs whose existence the objector wishes to deny. (The difference is only that logical techniques are topic-neutral, whereas we are discussing informal proofs concerned with special kinds of observable properties.)

7.D.9.a. This last point is important, because one of the strong motivating forces behind the desire to establish that all necessary truths are analytic, or that all apriori knowledge is knowledge of analytic truth, is the desire to eliminate the need to talk about special kinds of "insight" into the relations between universals. It is apparently thought that if all necessary truth and apriori knowledge can be shown to be derived by purely logical considerations from definitions then that need will be eliminated, but the remarks of the previous paragraph, and 5.C.9-10, show that this merely shifts the problem.

The amazing thing is that some philosophers thought this problem could be avoided by explaining all logical connections and all perception of logical connections in terms of formal systems and derivability of theorems from specified axioms according to rules of derivation specified in advance. It is amazing for two reasons, firstly because it is hard to see how anyone was ever able to think that merely talking about rules for manipulating symbols could explain logical properties and relations of statements (see appendix II), and secondly because even if talk about proofs in formal systems did explain logical connections and our knowledge of them, this would be at the cost of reducing logic to geometry, and there would remain the problem of explaining what sort of insight was involved in perceiving that strings of symbols stood in certain geometrical (or syntactical) relations to others. For, after all, the formal logician is not trying to establish the merely contingent fact that he can here and now derive (or has here and now derived) this particular set of marks from that particular set of marks (all tokens) while trying to follow certain rules: he wishes to show that a relation holds between types of marks, or, in other words, that geometrical properties (or patterns) stand in a connection of the kind which we have been discussing and he thinks he can explain away.

7.D.9.b. Sometimes it is argued that this question about the justification for asserting that a formula has a formal property need not arise (e.g. the property of being a theorem) since the rules of the formal system are so devised that even a moron, or a machine, could be instructed to check a proof to see that it went in accord with the rules. But this misses the point, for it relies on the very fact to which attention was drawn above (7.D.5), namely that each step in a proof must depend on connections between (e.g.) properties which are so evident as to need no proof.

If I "prove" that the statement "If a triangle is inside a circle, and a dot is inside the triangle, then the dot is inside the circle" is necessarily true, by drawing a diagram, namely a circle surrounding a triangle with a dot in the middle, or if a moron comes upon the diagram and utters the statement, then there is no more doubt about mistakes here than when a moron asserts that some pattern of formulae satisfies a recursive definition of "proof-sequence".



In any case, pointing out that a moron can apply some test for picking out certain sequences of sentences does not answer the question why sequences of sentences picked out in this way are valid proofs. Are morons supposed to be able to tell that any sequence of statements, no matter how complicated, which satisfies some recursive test constitutes a valid proof?

7.D.9.c. The assertion that an informal proof is really a formal proof in disguise does not seem to be of any help at all. It would be truer to say that a formal proof is an informal proof in disguise. The formulae in a formal proof <u>represent</u> propositions, or the logical forms of propositions, and serve the purpose of drawing our attention to logical relations between those propositions, in an indirect way. For the symbols are so chosen, that geometrical relations between them represent relations between the logical techniques corresponding to the logical forms of propositions (see 5.C.9). So, when we look at the logician's symbols, our attention is drawn (half consciously) to the relations between these logical techniques for discovering truth-values, and we are thereby enabled to see the relations between truth-values of propositions (i.e. relations between the outcomes of applying the logical techniques). It is not <u>essential</u>

to this process that the symbols which draw our attention to logical relations between propositions should constitute a "proof-sequence" in some formal system. (Herein lies the answer to Wittgenstein's puzzle in R.F.M. Part II sections 38 and 43: a Russellian proof is cogent only insofar as it has geometrical cogency, yet one may "accept" it without ever noticing the geometrical application. This is because the geometrical application is a consequence of the contingent fact that the rules of our language correlate logical forms of propositions, or logical techniques of verification, with geometrical forms of sentences in a uniform way.)

A fully explicit logical proof would draw attention directly to logical techniques and their interrelations, in much the same way as the informal proofs under discussion draw attention to connections between geometrical properties. Such an informal logical proof would help someone to perceive logical properties of or relations between propositions, which is what a formal proof does only indirectly and implicitly. (Compare Appendix II.)

7.D.10. We have now dealt with two objections, first that the account of informal proof was too psychological, and secondly that there is a formal proof, proceeding from definitions or identifying facts about meanings, underlying every informal proof. The first objection was met by asking what a proof is supposed to do, the second by showing that only an informal proof can do this, even if it is purely logical. But it is still open to someone who wishes to deny the existence of synthetic necessary truths to admit that informal proofs are possible, while asserting that they can only start from identifying relations between meanings or properties and proceed

<u>logically</u>, so that at every stage only topic-neutral considerations are relevant. Such a proof could only demonstrate the truth of an <u>analytic</u> proposition, according to the definition of 6.C.10. This assertion might be based on the argument that only an identifying relation between meanings can guarantee that there will not be an exception or counter-example to the proposition proved.

How can I be sure, simply because I have seen one rectangular figure divided up into a pair of triangles and a pair of parallelograms, that there will never be a rectangle which cannot be divided up in this way? How can I be <u>sure</u> that I shall never see a figure which has both the property of being round and the property of being square? The suggestion is that I can be sure only if I adopt a linguistic <u>convention</u> ruling out the possibility of describing any unexpected object as a counter-example to the proved proposition. That this suggestion has a point is shown by the history of mathematics. For it has happened more than once that a proof has been accepted as valid, and then later shown to be invalid. (Dr I. Lakatos has investigated such cases.) Many of Wittgenstein's remarks in <u>R.F.M.</u> seem to be directed towards showing that the possibility of an unexpected counter-example can <u>never</u> be eliminated except by a convention relating the meanings of words, in something like the way in which the "purely verbal" rules described in section 4.C were able to rule out the possibility of a counter-example to statements like "No red thing is orange" (see 4.C.3-4). (Similar, though more vague, arguments were used by von Wright on p.38 of "Logical Problem of Induction", 2nd Ed.)

7.D.10.a. Suppose, for example, that a (token) proof connecting a property P with a property Q purports to show that any object with P can serve as the starting point for another (token) proof of the same type. If so, the object must also provide an instance of the "bridge-property" starting with P and ending with Q, so it must be an instance of Q, and thus no counter-example to "All P things are Q" is possible. But suppose the proof does not work: an object turns up which has the property P, but from which the bridge-property cannot start, that is, an object which has P but cannot enter into a token-proof of the required type (the construction cannot be carried out). Then the only way to save the theorem proved is to adopt a new linguistic convention. E.g. we may say that being the starting point for the bridge-property is one of the defining criteria for having the property P, and so the new object does not really have P and is not really a counter-example. Now the argument we are considering claims that even before any counter-example has turned up, the only way we can guarantee that none <u>will</u> do so is by adopting this sort of new defining criterion for having some property, and what the informal proof does is show us how the new criterion works by displaying the bridge-property which we thereafter take to be identifyingly related to the property P. Similarly, we may have to take the connection between the bridge-property and the property Q as an identifying fact about the meaning of the word which formerly referred to Q, in order to ensure that no counter-example may turn up to break the link at the other end. (This applies only to one, relatively simple, kind of proof, of course.) By adopting these new rules we have given a more determinate meaning to the words expressing the theorem which was meant to be proved, and we have also set up identifying relations between their meanings from which it follows logically (see 6.B.11,ff.) that the sentence expressing the theorem in question must correspond to the truth-value "true". In Wittgenstein's terminology: "in the proof I have won through to a decision" (R.F.M. Part II, 27.)

7.D.10.b. Now it is very likely that this sort of thing happens sometimes in mathematics: we may think that we have completely identified some complicated geometrical or arithmetical property when in fact it is indeterminate in some respects (see section 4.A), and borderline cases could occur to provide potential counter-examples to some theorem about that property. (Cf. end of 7.B.4.) Then the construction used in a proof of that theorem may show us a new way of defining the meanings of the words used to express the theorem so as to rule out these counter-examples. But it is important to notice the differences between borderline cases which produce the following reactions (a) "I had not thought of that possibility, so I was wrong after all", (b) "I had not considered that possibility, but it does not matter as it is not the kind of thing that I was talking about" and (c) "I had not considered that possibility, and now I don't know whether to say it is the kind of thing I meant to refer to or not." Case (a) is an admission that the proof was invalid and the theorem wrong after all. (b) rejects the borderline case as not providing a counter-example. Only (c) leaves room for a new decision to adopt a convention as to how to describe the borderline case in order to save the theorem. The thesis that every proof is covertly a logical proof from identifying relations between concepts which we implicitly accept in accepting the proof requires that every geometrical concept be indeterminate in such a way that there is room for this sort of decision.

It is not possible to go into the question whether every concept is indeterminate in this way without embarking on a general enquiry concerning meaning and universals, and all the topics raised by Wittgenstein in his discussion of the notion of "following a rule" in "Philosophical Investigations". I shall say only that the fact that in some cases mathematicians have failed to see the possibility of counter-examples to propositions which they believed to have been proved, does not in the least convince me that no non-logical informal proof is secure, and neither does the fact that in some cases a property or a proof may be so complex that it cannot be "surveyed" properly and has some indeterminate aspects: there are other cases where properties are sufficiently simple for their connections to be quite perspicuous, leaving no room for any doubt that something will go wrong. I am perfectly certain that if anyone brings me an object which is alleged to be bounded by four plane surfaces, and not to have four vertices, then it will turn out that either the four planes do not bound the object, or there are not exactly four of them, or they are not planes, or he has miscounted the vertices, or .... (Why should I specify in advance all the mistakes which could possibly be made?)

7.D.10.c. I conclude that there is little reason to doubt that there are some connections between properties of such a kind as to prevent their occurring in certain combinations in particular objects, that these connections need not be identifying connections, and that they may either be quite evident, or sometimes made evident by an informal proof, which enables us therefore to see that some proposition states a necessary truth. This does not deny that there are cases where indeterminateness of meaning makes it necessary to adopt purely verbal rules (4.C) to rule out the possibility of counter-examples, neither does it imply that we are infallible and can never wrongly think we see connections between complicated properties.

It is worth noting that the difference between a doubtful borderline case of an instance of a property, and other objects which clearly are or clearly are not instances of it cannot be merely a numerical difference between them. There must be a difference in kind between borderline and non-borderline cases, they must exhibit different properties (e.g. an object with a borderline shade of red looks different from one which is bright scarlet). So even where a new verbal rule is required to ensure that borderline cases do not provide counter-examples to theorems proved by an informal proof, the verbal rule has to be applied only in some kinds of situations, involving objects which differ in certain respects from those which are not borderline cases. In other sorts of instances of the properties referred to, the connections between the properties are as shown in the informal proof, so no verbal rule is required to ensure that no counter-example to the theorem can arise amongst them, that is amongst objects which do not have the properties peculiar to the borderline cases. This shows that even if it is true that some verbal rule is always required to make it certain that no unexpected borderline cases can provide counter-examples to a theorem proved in the manner under discussion; that is, even if every necessary truth has an analytic aspect, nevertheless there is a synthetic aspect, brought out by the informal proof, which shows a necessary connection in at least a limited range of cases. ("No rectangle can look <u>just</u> like this one and fail to be divisible into two triangles and two parallelograms, and I do not need to adopt any convention to ensure that, for it is evident to anyone who examines the shape of this rectangle".)

7.D.11. Now our persistent objector may argue that even if it cannot be shown that what goes on in an informal proof is always implicit logical deduction from implicitly acknowledged identifying relations between meanings or properties, nevertheless it remains for me to demonstrate that the "connections" revealed by such proofs are not breakable, that they are <u>necessary</u> connections, allowing no exceptional particular objects as possible. How do we know that the constant conjunction of these two immediately perceptible features, or their failure to occur together, as the case may be, is not just a contingent fact? Certainly it is up to me to show that the propositions in question are necessarily true, but I do not do this by means of a philosophical argument, I do so by means of the proof which we are discussing!

If the objector cannot see for himself by examining properties (either in particular instances, or in his imagination if he is well acquainted with them), or by going through an informal proof, that nothing can be both round and square, that every tetralateral object is tetrahedral and no exceptions are possible, if he cannot see the necessary connections between these independently identifiable properties, then he cannot be forced to see them, as already pointed out, and he must forever remain in doubt as to whether, perhaps in the depths of darkest Africa, there lies hidden somewhere a slab of some hitherto unknown material, whose boundary can be seen to be at once both square and circular, or perhaps a little pyramid, completely enclosed by exactly four flat sides, but with only three vertices, or five.

The objector can surely not expect me to convince him by offering a proof which starts from definitions, and then draws purely logical inferences, for the whole point of this section and the last is to show that only by altering the meaning of the statement proved can one replace a non-logical informal proof by a logical one. (To show that the statements in question are necessarily true is not a philosophical task.)

7.D.11.note. All this must be qualified by the remark that our ordinary geometrical concepts ("round", "square", "straight", "flat", etc.) are extremely complicated in a way which makes it very difficult to describe clearly what goes on when a normal person is confronted with an informal proof of the sort used in school geometry. I am referring to the fact that where we have one word, such as "ellipse", there are usually very many concepts superimposed in its meaning, in a quite indeterminate way, and this is not noticed, owing to our use of loose criteria for identity of meaning, which cannot distinguish these different superimposed concepts. Consider each of the various definitions of the notion of an ellipse which might be given by a mathematician, and blur its edges a little. Add the semantic correlation between the word and the visual property or range of visual properties which we associate with it. Load all these meanings onto one word in an indeterminate way: and then try to explain what goes on in the mind of someone who uses the word in this overdetermined fashion when he sees a proof of a theorem about ellipses! (For other examples of superimposed concepts see 3.E.2 and 6.D.3. Compare also 5.E.7.a. I believe that our ordinary arithmetical concepts are indeterminate and overdetermined in a similar fashion, which is why different philosophies of mathematics have all been able to claim some plausibility - logicism, formalism, intuitionism, empiricist theories. Each has picked out one aspect of the truth, while making the mistake of claiming it to be the whole truth. There is no time now to show how a unifying theory could be developed.)

7.D.12. Some of the things said by Kant about synthetic apriori knowledge are explained by this discussion, in particular that it requires an "appeal to intuition". This is firstly because without the intuition (acquaintance with properties, etc.) one cannot know which concepts are involved or that there are any empirical concepts involved which can be applied to observable objects; and secondly an appeal to intuition is required since without it one cannot come to see how the concepts (or properties) are connected. (See, for example, C.P.R., A.239-40, B.298,ff; B.308-9; A.716,B.744). The fact that looking at a diagram (real or imagined) can play an essential part in perceiving the connections between properties shows that in doing so one is not merely drawing logical inferences whose validity depends on topic-neutral principles. (This might also be put by saying that the type-proof, or bridge-property, mentioned in 7.D.4, is not logically synthesized out of the properties whose connection it is supposed to reveal: the synthesis in the proof - i.e. the way in which it is constructed - is non-logical, for reasons of the sort given in 3.D.6,ff.)

I think my account of informal proof helps also to explain why Frege believed that one could have synthetic apriori knowledge concerning geometry. (See sections 14 and 90 of "Foundations of Arithmetic".) It also explains some of the talk of Intuitionists about "mental constructions" (see Heyting "Intuitionism", e.g. p.6,ff), though my account would have to be modified to take account of proofs of theorems about the properties of infinite sets, since these are not perceptible properties.

7.D.13. Whether this brief and highly condensed sketch is correct or not, one thing should be clear: informal proofs certainly do something, and what they do is different from what is done by a proof of an analytic proposition starting from identifying facts about meanings and proceeding logically. This shows that if the "proved" statements are necessarily true in both cases, then it is very likely that there are at least two kinds of necessity worth distinguishing. My suggestion is that the way to distinguish them is to notice that in both cases the propositions exemplify the notion of a "freak" case of a rogator whose value happens to be determined by relations between its arguments together with facts about the technique for working out its values, the difference being that in the one case the arguments are identifyingly related, and all arguments standing in the same relation must yield the same truth-value, whereas in the other case the relations between the arguments are not identifying, and they are not relations of a sort which could hold between any sorts of entities at all (they are not topic-neutral).

However, it is open to anyone who does not like talking about synthetic necessary truths or synthetic apriori knowledge to reject my terminology and say that in both cases the propositions are analytic since their truth is determined, however indirectly, by what they mean. But then "true in virtue of meanings" seems to be synonymous with "necessarily true" in this terminology, and the assertion that all necessary truths are analytic says nothing and says it in a redundant terminology which fails to take account of distinctions which are of some interest.

7.D.14. It should not be thought that the assertion that there are synthetic necessary truths has any great metaphysical significance, or that it justifies any claim to have perceived with the inner light of reason, or any other mysterious faculty, moral or theological truths, or truths of a transcendent nature. (See 7.D.8). So far, the assertion has been justified only by a discussion of ways of perceiving connections between simple empirical statements (i.e. between the techniques corresponding to logical rogators). If it can be extended to cover other cases, such as the principle of causality, then this has to be shown by detailed investigation.

I claim only to have given an <u>informal</u> proof of the existence of synthetic necessary truths of a simple and uninteresting kind, or at least to have shown that there is a distinction to be made between different sorts of necessary truth. But the topic is difficult and complex, and I have been unable to do much more than provide an introduction to it by showing how its problems are related to and can arise out of general considerations about thought and language and experience. [I am very dissatisfied with the discussion of this section, though I believe it to be a first step in the right direction. I have included it for the sake of completeness.]

7.E. Additional remarks

7.E.1. This chapter may now be summarized. Chapter six had explained how the truth of a statement may be a purely logical consequence of identifying facts about the meanings of the words used to express it. Such a statement would be true in any possible state of the world because its truth-value is determined independently of how the world happens to be. In this chapter I have tried to explain why it makes sense to contrast the way the world happens to be with ways it might have been, so as to give a fairly clear sense to the question: Are there any statements which would be true in all possible states of the world, besides those described in chapter six? That is: are there any non-analytic necessary truths? I was able to give a sense to this question, by making use of some of the very general facts about our language which were pointed out in chapter two, especially 2.B and 2.D, namely the fact that we use a conceptual scheme with provision for a distinction between universals (observable properties and relations) and particulars, and the fact that universals are not essentially tied to actual particulars. The question then became: Are there any limitations on the distribution of universals to be found in any actual or possible state of the world, apart from purely logical limitations, which are in no way concerned with anything special about specific properties but are topic-neutral? (This was the fundamental question, but in order to take account of "improper" or "synthesized" universals (see chapter three), we asked the question in the following form: Are there any connections between universals (i.e. restrictions on the ways in which they may be instantiated) which are not due simply to (a) identifying facts about those universals and (b) purely logical or topic-neutral restrictions?]

7.E.2. I tried to answer this question by drawing attention to observable connections between observable properties, where (a) the properties can be identified independently of each other and (b) logical considerations alone do not account for the connections between them, since the properties themselves must be examined for the connection to be perceived. Thus, a slightly more general account was available of the way in which relations between the arguments of a logical rogator might help to determine its value, than the account given in chapter six. In short, we saw that a sentence may express a proposition which would be true in all possible states of the world, though it is not analytic. The reason why no exceptions to such a synthetic necessary truth are possible is that exceptions would have to be objects in which properties were combined in ways which are excluded by the connections between those properties.

7.E.3. This also explained why one might have the right to make such statements as "If this had been P then it would also have been Q" or "If this had been P then it would not have been R". The connections between properties which make some statements necessarily true, also give a sense to counterfactual conditional statements, by giving us the right to assert some of them as true.

(Note: We could generalize this slightly to explain the concept of entailment. The proposition p entails the proposition q if something ensures that if p were true then q would be. This can be put more precisely:

The proposition p entails the proposition q if and only if there is some relation R satisfying the following conditions:

1) The relation holds between p and q.
2) The relation holds between some propositions which are neither necessarily true nor necessarily false.
3) If the relation holds between two propositions Ø and Ø, then this ensures that in any possible state of the world in which Ø would be true Ø would also be true.

(The relation may be purely formal - i.e. it may be a topic-neutral relation, or it may be concerned with the content of the two propositions.)

I suspect that our ordinary expressions of the form "If Ø then Ø" are more like assertions of entailment than like assertions of material implication, though probably much more complex than either, as can be seen by examining the sorts of things which are normally regarded as justifying such assertions. In consequence, it is not obvious that what I said in chapter five about logical forms of propositions, and the logical rogators which correspond to them, applies without modification to conditional statements.)

7.E.4. The discussion of informal proofs was intended to explain how we can become aware of connections between properties of the sorts which ensure the necessary truth of some synthetic propositions. It also provided a partial answer to the question raised in section 5.C about the manner in which one can become aware of the relations between logical techniques in virtue of which propositions whose logical forms corresponded to those techniques might have logical properties or stand in logical relations. The answer was very vague, namely that perceiving connections between logical techniques is the same sort of thing as perceiving connections between (say) geometrical properties.

It is clear that there is a lot more work to be done on the subject of informal proof, as I have talked only about some very simple cases, and left many questions unanswered.

7.E.5. For example, there is a puzzling fact which I have hardly mentioned, except in 6.E.6-7, and without an explanation of which it is impossible to give a complete account of the way in which informal proofs work, or the way in which we normally come to have knowledge of necessary truth, namely the fact that a person may be justified in claiming or believing something, and for the right reasons, without his being able to see clearly or say clearly what the reasons are.

This is exemplified by a layman's assertion of an analytic proposition which he correctly justifies by saying that it is "true by definition", even though he may be quite unable to explain what it is for a proposition to be true by definition. Similarly, he may have seen an informal proof, and so be quite justified in asserting the proposition which it proves, saying "It must be so", and yet be quite incapable of saying <u>how</u> the proof proves the proposition. This is connected with some of the remarks in the appendix on "Implicit Knowledge".

It is pretty certain that if ever a philosopher does manage to give a clear and accurate account of how informal proofs work and why we are justified in asserting the propositions which such proofs are taken to justify, we shall not be able to retort to him that we knew it all before, just as the person who cannot see that some statement is true until he has studied a proof cannot claim to have known it all before, even though the proof does lead him on from things which he did know before. Perhaps there is an analogy here between what happens when a mathematician convinces us of the truth of some surprising theorem by drawing construction-lines and what happens when a philosopher solves some kinds of problem: the philosopher draws "construction-lines" of a different sort in order to bring out connections between concepts, such as the connection between the concept of a diagram used in a geometrical proof and the concept of necessity. (Was I drawing philosophical "construction-lines" when I talked about rogators in chapter five, in order to give an account of logical form and explain the connection between formal properties of sentences and logical properties of propositions?) This suggests that if a mathematical proof can enable one to see that some synthetic proposition is necessarily true, then perhaps philosophical investigations may also reveal synthetic necessary truths. This is something which requires detailed investigation. (For some remarks on philosophical analysis, see Appendix IV.)

7.E.6. Another subject which requires investigation is the relation between our ordinary empirical concepts of shape and colour, and the idealized concepts which, at times, it may have appeared that I was discussing. (See the disclaimers in 3.A.2. and 7.C.7.) Idealized concepts are somehow extrapolated from our ordinary concepts. Examples are the concept of an "absolutely specific" shade of red, or an "absolutely specific" triangular shape, and the concept of a "perfect" geometrical shape, such as the shape of a perfect cube. Philosophers sometimes suggest that there is no connection between these idealized concepts and our ordinary empirical ones. At any rate they are usually unclear as to how they are derived from our empirical concepts, perhaps because they fail to see that two sorts of idealization are involved.

7.E.6.a. First there is the idealization towards perfect specificity, which explains our use of expressions like "exactly the same shade of colour as" or "exactly the same shape as". We see pairs of objects which are more and more alike in some respect, and then extrapolate to the limit, on the assumption that it makes sense to do so, even though we are not able to discriminate properties finely enough to base the notion of "exact likeness" directly in experience. This kind of exact likeness is supposed to be transitive, unlike perceptual likeness. So the absolutely specific shade of colour (for example) of my table is a property common to all objects exactly like it in respect of colour. (There can be no borderline cases.) Perhaps some argument can be given to justify the assumption that it makes sense to talk like this. I do not know.

7.E.6.b. The second sort of idealization is quite different, and helps to explain such concepts as the concept of a "perfectly straight" line, or a "perfectly smooth" curve, or a "perfectly plane" surface, or a "perfectly perpendicular" pair of lines. It may also be connected with such notions as a "perfectly pure" shade of red, or a "perfectly pure tone", or a "perfect" musical octave. For the purposes of this sort of idealizations it is first of all <u>assumed</u> that it makes sense to speak of absolutely specific properties (shapes, colours, tones, etc.) and then use is made of the fact that in some cases the properties can be arranged in a series, apparently tending towards a limit. Thus one line looks straighter (smoother, more nearly circular, etc.) than another, and a third looks straighter (etc.) than the first, and so on: so we extrapolate and assume that it makes sense to talk about the perfectly straight (smooth, circular, etc.) line which lies at the end of the series and is straighter (smoother, etc.) than the others. A similar process may account for the concept of an infinitely long line, or the concept of perfectly parallel lines, or the concept of infinity in arithmetic or set theory. Similarly, one colour looks a purer blue than another, and so on, so we assume that there could be a perfectly pure shade of blue. In some cases there may be more than one route by which the limit is approached.

7.E.6.c. It is taken for granted that such methods of extrapolation fully define the "perfect" concepts which they generate, and that different methods of extrapolation may define the same limit. And it used to be thought that facts about these perfect concepts could be discovered with the aid of old-style Euclidean proofs. But it is more likely that although the method of generation of such idealized concepts fully determines some things about them (thus, the relation "inside", applied to perfect squares, triangles, circles, etc., is transitive), nevertheless in order fully to define them it may be necessary arbitrarily to stipulate that certain relations hold between them, or that certain statements about them are true (such as Euclid's parallel axiom). Since such a stipulation is arbitrary (there is nothing in virtue of which it is "correct"), we could adopt alternative "axioms" and complete the definitions in another way. This is the tiny grain of truth which lies behind current opinions of the sort which I criticised in 7.C.7(note).

7.E.6.d. It is also sometimes not noticed that the process of idealization does not remove all empirical elements from these "perfect" concepts. Hence it is assumed that geometrical proofs which are concerned with them have nothing to do with objects of experience. This is why philosophers sometimes talk as if a perfectly sharp distinction can be made between "pure mathematics" and "applied mathematics", the latter being regarded as an empirical science, perhaps a branch of physics. There is no space here to explain in detail why this is muddled.

7.E.7. A failure to understand the nature of these "perfect" mathematical concepts, or to see the difference between those "axioms" which served the purpose of completing the definitions of concepts and the "theorems" whose truth in no way depended on arbitrary stipulations of identifying conventions, left people unable to cope with the shock of the discovery of alternative internally consistent axiom-systems for geometry. The notion of a proof as something which served to establish the <u>truth</u> of a theorem was therefore undermined, and philosophers tried to salvage what was left by treating proofs as nothing more than methods of deducing consequences from arbitrary hypotheses or postulates. This at least seemed to be secure: for, by means of formalized systems of logic one could at least give fool-proof criteria for the validity of a proof. Criticisms of this conception of proof have been made elsewhere (in 5.C.10,ff, 7.D.9,ff and Appendix II). It seems not to have been realized that such a conception severs the concept of "proof" completely from the concept of "truth". It seems not to have been realized that if proofs are intended to serve the purposes described in section 7.D, namely, to enable people to perceive the truth of propositions, to bring out the reasons why propositions are true, then the search for a fool-proof criterion of validity is futile: for, no matter what criterion is adopted, questions remain about the justification for accepting proofs which satisfy that criterion, and the justification for the statement that any particular type of proof satisfied the criterion. If a justification is offered, then <u>its</u> validity cannot be constituted by satisfaction of the criterion in question - that would be circular. The only way to avoid a circle is to give up talking about criteria of validity, and either follow Wittgenstein in his talk about conventions (in <u>R.F.M.</u>) or try to explain how we can simply <u>see</u> necessary connections between properties and other universals by examining them, perhaps with the aid of informal proofs. (Are these really distinct alternatives?)

7.E.8. Finally, the reader is reminded that although an informal proof enables one to discover that a proposition like "All tetralaterals are tetrahedrals" is true without discovering how things happen to be in the world (i.e. without looking to see which particular objects exist where, and what properties they have, etc.), nevertheless it is possible to verify such a proposition empirically, just as (cf. section 6.E) it is possible to verify an analytic proposition empirically. Thus, one might carry out a survey of all objects bounded by four plane sides in order to discover whether they also possess the property of having four vertices. Such an empirical justification for the assertion of the proposition is adequate, despite the fact that it is unnecessary.