
(DRAFT: Liable to change)

Turing Conversation Draft

Kant, Turing, Chemistry and
Challenges to Current Theories About
Brains, Minds, and Forms of Computation.

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This document is temporarily at
<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/turing-conv.html>

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The Question

How did biological evolution produce brains that can make mathematical discoveries about spatial structures, e.g. ancient discoveries in geometry, that cannot be explained by known brain mechanisms, and are not matched by anything in current AI?

I'll give examples of forms of spatial intelligence in humans (including very young humans) and other animals that are not explained by current neuroscientific theories, and cannot be achieved by computer-based AI systems, e.g. logical theorem provers or neural-net based systems, which are inadequate for different reasons. I'll then outline a long term research project to discover the required brain mechanisms by investigating aspects of biological evolution, starting from the earliest forms of life, involving increasingly complex chemical control functions required for processes of development of organisms from a fertilized egg onwards. I suggest that close investigation of those assembly mechanisms, and the control problems they have to solve, will eventual reveal forms of reasoning based on chemical mechanisms that have played essential roles in evolution since long before there were animals with brains. I'll suggest (though I cannot yet prove) that the mixtures of discrete and continuous processes in chemical mechanisms support forms of computation that cannot be achieved by digital computers. Moreover, despite the power of deep learning mechanisms they derive probabilities from statistical evidence but are constitutionally incapable of discovering, or even representing geometric impossibility or necessity, which were features of ancient mathematical discoveries leading up to Euclid. Moreover, even young human toddlers and other intelligent species (e.g. squirrels) can make use of impossibility, and in some cases reasoning about necessity, in making choices between actions and action sequences. More sophisticated versions of those capabilities were used by ancient geometers, and are still essential in engineering design and other activities requiring human intelligence, e.g. driving a car through cluttered narrow streets in busy parts of small towns, as opposed to driving on motorways.

Introduction

This very personal "conversation piece" introduces a collection of topics clustered around, and branching out from, Alan Turing's work. My comments are also partly inspired by Immanuel Kant's views on mathematical knowledge, a century and a half before Turing expressed apparently related views on mathematical intuition. I shall try to explain, at least schematically, the relevance of developments in molecular biology, a field mentioned by Turing in his paper on morphogenesis ([1952](#)) and hinted at in his reference in ([1950](#)) to the importance of chemistry in brains. I suspect he thought chemical interactions played a role in mechanisms making mathematical intuition possible, which he had contrasted with mathematical ingenuity in his thesis: he thought computers could have ingenuity but not intuition. I'll give reasons for thinking that sub-neural chemical

processes may be able to go beyond what digital computers can do -- thereby supporting Turing's comment, especially if sub-neural chemical mechanisms turn out to play a role in ancient forms of mathematical consciousness, including consciousness of mathematical necessity or impossibility.

In the last few decades there has been a vast amount of research in many fields of biology, accumulating potentially relevant information about sub-cellular chemical mechanisms, including pathogen recognition in the immune system, production and use of antibody molecules, mechanisms of gene expression, growth, tissue repair and many more.

Since the discovery of the structure of DNA in mid 20th Century, constantly accelerating work on chemical mechanisms of gene expression has enormously expanded our understanding in many fields (including developmental-biology, medicine, immunology and neuroscience). I suggest that some of these discoveries have the potential to support Turing's claim in 1950 that *"In the nervous system chemical phenomena are at least as important as electrical"*. He seems there to be hinting that molecular processes within neurons are as important (for mathematical thought processes?) as electrical signals between neurons, though he did not say why, or in what way. This paper supports that claim, and tries to draw out some important consequences that have gone unnoticed, including relevance to Kant's claims about mathematical knowledge. However, I shall not propose specific chemical mechanisms.

Kant's view of mathematical discovery has been widely criticised (and more recently ignored), in my view mistakenly, on the assumption that Euclidean geometry, the source of some of Kant's examples, had been refuted by a combination of discovery of alternative geometries, Einstein's theory of general relativity, claiming that physical space was not Euclidean, and Eddington's confirmation of Einstein's work, based on observations of the solar eclipse in 1919. However, as I tried to show in my [1962 DPhil Thesis](#), those developments did not change the *fact* that ancient mathematicians had discovered, and explored, the mathematical structure now labelled "Euclidean Geometry", including making deep discoveries about it that were neither empirical discoveries nor merely logical consequences of definitions, as Kant understood.

I shall try to show below that there is an unobvious connection between Kant's characterisation of such discoveries and Turing's scant and obscure remarks on intuition vs ingenuity. The topic was discussed inconclusively earlier, in <https://www.cs.bham.ac.uk/research/projects/cogaff/misc/turing-intuition.html>.

Although science and philosophy differ in their details, there is no sharp boundary between them, as explained in Chapter 2 of [Sloman\(1978\)](#). What may start off as an attempt to answer a philosophical question, e.g. about the nature of space, or of life, can, over time, spawn important new science, without any sudden transition. Both science and philosophy are concerned primarily with the investigation of what is possible and how it is possible, and also which combinations of possibilities are impossible, and why. So scientific research is not just, and not even primarily, a search for reliable predictive laws: its aims are much deeper than that. In contrast, activities concerned with use of statistical evidence to derive *probabilities* are much more shallow, although when practical decisions have to be taken they may be better than nothing, sometimes much better, e.g. betting on tomorrow's weather.

Can a baby robot grow up to be a mathematician?

In the 1970s, having begun to learn about AI and programming, I hoped to build a working model illustrating Kant's ideas, by showing how a suitably pre-programmed Kantian "baby" robot could grow up to be a mathematician, like the ancient geometers.

For various reasons, designing (let alone building!) a "baby" robot mathematician proved very difficult. Four decades later, events related to Turing-centenary celebrations and an invitation to contribute a number of papers to a commemorative volume [Cooper et.al\(2013\)](#), triggered a new stream of ideas, growing steadily since then, including launching the Turing-inspired Meta-Morphogenesis project [\(2013\)](#), which investigates (recursive) evolution of mechanisms of biological evolution and development. This project continues to spawn a collection of related interacting sub-projects -- all still "work in progress" (or "unfinished business"), unfortunately.

The sub-projects include a still developing theory of evolved biological construction kits, including construction kits for building new construction-kits, in processes that repeatedly add new types of information, new types of control, and new types of causation to the contents of the universe (or at least our part of it). Moreover processes involved in evolution repeatedly extend the complexity of the types of information used by mechanisms and processes of evolution -- because new forms of life make use of previous forms of life and information about previous forms.

A similar kind of recursive(?) layering of metaphysical types, and information about metaphysical types occurs in the development of complex, self-extending, self-maintaining, organisms.

The task of understanding all this is horrendously complex, partly because of the complexities of biological mechanisms, and the varieties of development of individual organisms, across scales of size and complexity. This includes evolution and development of the mathematical competences discussed by Kant.

Evolution of mathematical abilities

I shall try to summarise some (conjectured) aspects of precursors and origins of mathematical abilities in early products of evolution, including conjectures regarding requirements for very early control mechanisms involved in sub-cellular chemical processing, which need to use information about the context in selecting actions. These ancient control mechanisms, in use long before the evolution of nervous systems and brains, are based on chemical processes that depend on features of quantum mechanics, in the manner outlined by Schrödinger's *What is life?* [\(1944\)](#), which naturally leads to new lines of thought about the roles of chemical information and chemical information processing in many parts of biological organisms. Those roles are likely to be important for growth, production of new control processes, and many physiological functions, dependent on the quantum mechanisms discussed by Schrödinger. I suspect Turing had already thought about some of those roles in 1952. I don't know whether he ever read Schrödinger's little book.

Although there is much still to be learnt about the earliest life forms, Tibor Ganti, using knowledge available by the 1970s, produced a specification for the simplest possible self-sustaining, self-reproducing life form, a single cell, the "Chemoton", that could survive and reproduce in a suitable chemical soup. The ideas are usefully summarised and reviewed in Korthof[\(2003\)](#). Such an organism requires a surprisingly complex collection of different components, all of which need to be maintained during normal life and duplicated during reproduction. Somehow, through the development of these mechanisms, evolution "discovered" powerful ways to compute (including

controlling) with chemistry.

Later, after brains had evolved, I suspect new uses of sub-cellular (i.e. sub-neural) chemistry were found, for production and control of cognitive processes in more complex organisms that evolved later, making use of brains. These sub-neural mechanisms are ignored by many neuroscientists and psychologists. The theories most of them consider in attempting to explain human mathematical competences generally ignore the complexities mentioned below, including abilities to discover and reason about impossibility and necessity: instead they normally focus only on a shallow subset of competences, e.g. simple numerical competences, amenable to reproducible laboratory tests, which rules out use of important evidence from individual behaviours. (Jean Piaget was an outstanding exception, as explained below.)

These ideas all indicate requirements for increasing complexity in the mechanisms explaining human mathematical competences, including the roles of evolved construction kits, and evolution's use of compositionality.

Both genetically informed developmental mechanisms, and the increasing complexity of information used by those mechanisms shows a need to understand in much more detail the roles of sub-cellular molecular information processing mechanisms, including mechanisms that make use of information from multiple sources in selecting goals, generating options for achieving goals and selecting and acting on those goals. (I use the word "information" in the sense of Jane Austen, not Claude Shannon, as explained in Sloman [\(2013--2018\)](#).)

The need for new levels of informed control

The information about current state and what needs to be done next, used by molecular construction processes, and much later for action-selection and execution processes, cannot be information about the basic underlying physics of the brain, including huge numbers of fundamental physical particles, since each increase in physical complexity vastly expands the number of fundamental particles involved. During all stages of development there are new requirements for control, including control of internal processes, such as goal selection and planning, and control of physical actions. Although the selected and controlled processes involve changes at molecular levels in brains and bodies, the control process cannot include information about all the sub-atomic physical particles involved in those processes, any more than a human controlling a dockside crane can reason about all the atoms, and sub-atomic particles of which the crane is controlled. Instead intelligent control needs to consider only "macro-level" changes, e.g. of spatial orientation of the crane, the angle of slope of a movable jib, changes of amount of cable to the lifting grabber to be raised or lowered, and when to release items moved to new locations.

This decomposition of control information is possible because so many parts of such a machine have structures in which relations between constituent molecules or sub-molecular particles, do not change. So the states of those parts can be represented by a few "high level" physical parameters in the mind of a human controller.

I suggest that long before human engineers produced such controllable complex mechanisms, biological evolution discovered and used the same principles. So control mechanisms in brains need consider only relative large portions of physical reality and their changing relations during performance of actions.

Similar simplification of control is a standard feature of a huge variety of human-designed controllable machines. Steering a car would be an impossible challenge if we had to use information about all the fundamental physical particles in the car and in the environment to decide how to change all their relationships. Instead we can select a few important "high level" relationships to be monitored and a few parameters to be controlled, e.g. acceleration/deceleration and direction of motion, and possibly also gear selection.

Evolution, like human control engineers much later, must have discovered how to build control mechanism that operate on information at a *tractable* level of complexity. The standard way to achieve that is to use *layers* of control: something evolution must have discovered long before we did.

In all organisms, there are changing layers of control during individual development and later during performance of "macro" biological functions, such as growth of roots, production then shedding of leaves, production then of reproductive mechanisms, and many more. For mobile animals the control processes require more high level integrated information and control mechanisms: if an animal is to move toward a food source, or away from a predator, a very large number of body parts, including muscles, need to have their behaviours coordinated, in the light of changing spatial relationships between the animal, or parts of the animal, and parts of the environment, which may include other animals (food, predators, conspecifics, mates, etc.) as well as paths, obstacles, and other features of the environment.

Not only is it not possible for such an animal to obtain information about all the molecules, or sub-atomic particles, in the environment or in its body, the task of taking decisions and controlling actions at that level of detail would be completely intractable.

None of that will surprise any readers. But my point is that similar discoveries must have been made over and over again by processes of biological evolution, including evolution of the smallest (e.g. single celled organisms), and even evolution of mechanism for assembling sub-microscopic parts of organisms during development of a fertilised seed or egg, as a new organism grows -- whether the new organism is a microbe, a plant (including giant trees), a boneless animal, a vertebrate, and so on.

That implies that at some distant stage in the past, mechanisms must have evolved that enabled the use of tractable *derived* information structures during control processes in which decisions are made -- e.g. when growing, or repairing new sub-cellular structures, or disposing of waste products, or triggering new processes (e.g. muscular contractions), or

So, long before humans started thinking about the nature of reality, and its levels of description, and producing notations for control and communication, all sorts of biological mechanisms must already have achieved such results and used them in control processes. One feature of evolution must then be continually adding new information layers as systems being controlled grow more complex (including control of controllers, etc.)

If all that is correct, it would be very surprising if the powerful sub-cellular mechanisms of control developed for production and assembly of parts of organisms were not also used for the control of behaviours of whole organisms, aided by development of information channels so that requirements detected in one part of the organism can be conveyed to other parts producing or controlling distribution of relevant resources or production of actions to change relationships to

parts of the environment, e.g. approaching food, avoiding obstacles, controlling manipulators like jaws or hands, all using information gained from multiple sensors.

We should now try to identify ways in which those ancient chemistry-based control mechanisms might still be used in brains of intelligent animals -- using sub-neural molecular mechanisms for many complex control decisions.

[Simple reflexes don't need so much complexity. Complex reflexes, e.g. in an expert racing driver, almost certainly do. Likewise sight-reading music.]

On this view, the main original function for neural networks may have been to convey information to be used by chemical (sub-neural) information processors.

I SUGGEST THAT A LATER DEVELOPMENT, WHOSE DETAILS ARE STILL COMPLETELY UNKNOWN WAS TO USE CHEMICAL MECHANISMS TO ANTICIPATE DEVELOPMENTS BY REASONING ABOUT POSSIBLE AND IMPOSSIBLE STATES AND CONFIGURATION CHANGES. THEREBY PRODUCING THE BRAINS OF ANCIENT MATHEMATICIANS LIKE ARCHIMEDES, EUCLID, ZENO AND MANY OTHERS. HOW COULD THAT HAVE HAPPENED?

This is very different function from the use of information to take decisions on the basis of learnt statistical regularities. If that were all that neural networks did that would leave organisms incapable of making ancient mathematical discoveries, e.g. about geometric and topological impossibilities and necessary connections.

Statistics-based mechanisms are also unable to account for the intelligence of squirrels, crows, and human toddlers who can discover and use spatial impossibilities and necessities, without being able to reflect on or communicate them. [And lots of other intelligent animals...]

These ideas were partly inspired, over several years, by the combined influences of: Kant, Turing's distinction between mathematical intuition and mathematical ingenuity in his PhD thesis (1938), a tantalizing comment in his *Mind* paper (1950), about the importance of chemistry in brains, his paper on chemistry-based morphogenesis (1952), and Schrödinger's ideas mentioned earlier, about the essential role of quantum mechanisms in biological reproduction. As far as I know, Turing never elaborated on his comment about chemistry in brains, perhaps because the thoughts were insufficiently developed. I'll try below to indicate some ways they could have been developed.

I'll also include selections from a large and still growing collection of research themes and results, linking life and its vast and complex chemical basis, since the discovery of the structure and functions of DNA reported in 1952, and subsequently. As I am not a specialist in any of those fields, I must apologise for errors and omissions, hoping that the key messages will not be seriously affected. As far as I know, researchers on structures and functions of DNA have not addressed these issues.

Biological organisms can have competences that develop before, or at various stages after, birth or hatching, but are not simply a result of training or learning and are not fully, explicitly, specified in the genome. What develops can result from specialised inherited mechanisms that are able to extend themselves by interacting with the internal or external environment, including conspecifics, such as the inherited (multi-layer) mechanisms that allow a typical human to learn any one of several thousand very different languages in appropriate environments, where the languages differ at many levels from the lowest level sound or signed patterns or written/printed patterns through

multiple linguistic levels, e.g. vocabulary, syntactic forms, semantic contents and pragmatic functions -- all acquired under the control of a shared multi-layer genome.

This has been labelled the "Meta-Configured Genome" hypotheses (MCG), referring to genome layers expressed at different stages of development, each layer building both on details that have already been developed and previously evolved abstractions used for organising the details. This phenomenon is particularly clear in the development of human language competences, including acquisition of sign languages in deaf communities.

The MCG theory is still under development as explained in Chappell, Sloman Tino ([2007-2019](#)). As applied to language developing the key MCG idea is that human languages are not so much *learned* as collaboratively *created*, using an extraordinary multi-layered collection of mechanisms specified in the genome, and produced at different stages of development, creating new layers of linguistic structure and function, built on products of previous layers within the individual and features of the environment that can vary from culture to culture, generation to generation, and also across physical environments. Often, but not always, the relevant environmental details are products of ancestors of the individuals.

When the youngest collaborators are constrained by the need to collaborate with a relatively large established community, the process may *appear* to involve only learning, not creation. The ability of deaf children in Nicaragua to develop their own new sign language reported in this video <https://www.youtube.com/watch?v=pjtiolFuNf8>. is a spectacular example of the MCG at work in sign-language *creation*.

It is striking that all the genetic mechanisms that are now involved in human language acquisition/creation almost certainly evolved before any language that is used by humans now: those languages are all too young (unless there is some undiscovered, isolated, community using a language that evolved in prehistoric times and never changed).

Could there be a similar genetic mechanism for producing physical parts (e.g. limbs) that vary as much as languages do? I think the answer is No! That's partly because most of the materials used in building physical structures are not as re-combinable as the components of languages: abstract patterns built from simpler patterns, where the patterns may be instantiated in different physical formats: spoken, written, signed, typed, flagged in semaphore signals, etc.

Many individuals learn more than one language, which requires complex divisions of storage space for different levels of linguistic information for different languages. So the human genome supports multi-layer, multi-branching, versions of the same type of competence, sometimes within a single individual fluent in several languages. There are many other multi-layer competences supported by the human genome, such as musical competences, social competences, agricultural competences, and hunting competences, that vary across cultures. In contrast, no amount of training and learning will teach any known non-human animal to report complex adventures in English, or to compose fugues in the style of Bach, although many have the ability to survive in a wide range of physical environments. (There are species, including parrots and chimpanzees that can pick up *fragments* (in some cases surprisingly rich fragments) of human linguistic competences, though as far as I can tell no parrot is able to invent a multi-sentence story.

Parallel Meta-Configured Development

Learning multiple languages involves acquiring, storing, and using specifications for multi-branching abstract structures that can generate linguistic behaviours. In contrast, generating multiple alternative branching types of body-part would not produce a viable organism, with multiple branching types of legs, wings, stalks, mouths, teeth, wheels(?), and also internal organs, e.g. stomachs, lungs, hearts, colons, etc. etc. This is an important difference between morphogenesis of *physical components* and morphogenesis of *competences*. (I don't think Turing explicitly mentioned that difference in his paper on morphogenesis, though I suspect he would have found it obvious.)

There are some commonalities between the above features of genetic specifications for language development and genetic specifications for mathematical achievements. In both cases the possible variants of the competences produced by the genome are far more numerous than the variations in body parts, such as arms or legs. This is partly because the genetic potential for linguistic development and for mathematical development both allow huge amounts and kinds of variation, unlike the genetic potential for numbers of instances, and shapes and capabilities, of types of major body-parts, such as arms, legs, moths, eyes, etc.

Individual variation shows that linguistic competences are not fully specified in the genome, yet there are fairly clearly defined *layers*, produced at different stages of development, most noticeable in linguistic competences, but also in other competences that can be (partly) shaped by an individual's environment during development. Examples of multi-layered competences, with increasing variation in later developments, include musical competences, social interaction competences that vary across cultures, mathematical competences, and many competences involved in manipulating physical objects that can vary across geographical environments, across historical epochs in the same location, and can also vary in the difficulty of acquisition, indicated by the amount and type of adult help or education required, e.g. competences involved in playing musical instruments.

Many human abilities to acquire new competences depend on internal species-specific biological mechanisms required for increasingly sophisticated chemistry-based forms of control during development of particular competences. These mechanisms support evolution of increasingly complex organisms using increasingly complex evolved information-processing capabilities, especially in mobile organisms that require some understanding of space (whether very primitive or highly sophisticated) using abilities to coordinate motions of body-parts to achieve new goals. System-wide control requirements for immobile plants are much simpler: roots and tips of branches can choose directions in which to grow without having to reach agreement on a direction.

The development of an organism, even a microscopic organism, from a fertilised egg requires a complex collection of sub-microscopic mechanisms using spatial manipulation of atoms and molecules. Manipulation is different from a process like stirring or shaking of a collection of parts: it requires bringing individual components together in ways that allow them to combine to form new substances and enduring structures. Controlled manipulation involves using information available at different stages to determine what to do when. We are all aware of humans and other animals, and gardeners are aware of plants, selecting between different options in different circumstances. We now know that this also happens invisibly, in processes of sub-microscopic molecular manipulation during transcription of complex molecules, either to make new copies of those molecules or to produce by-products specified by genetic information in those molecules. Some

examples, including molecular machines, assembling new molecules from old, are presented in a video tutorial by Peter Hoffmann(2012). So a form of primitive robot-like intelligence is involved in the most basic assembly processes during development of an organism from a fertilised egg or seed. Such sub-cellular intelligence must have existed on this planet long before there were any humans.

Closely related ideas are now being used in *human-designed* molecular, biomedical mechanisms for use in nanoscale medical machines, as explained by Daniela Wilson(2020), discussing "nano and microscale 'man-made' machines as autonomous systems with life-like behaviours", but without mentioning that long before the advent of human-designed molecular intelligence, biological evolution had produced and made use of multiple varieties of *natural* molecular intelligence, as explained below. This extends the discussion of evolved varieties of consciousness in Sloman(2020).

Biological reproductive, developmental and maintenance processes in all organisms require an enormous amount of chemically controlled chemical synthesis of new materials and components, and also chemically controlled low-level decision making, e.g. about how to disassemble complex molecules in food, where to send the resulting components, what to grow where, how to provide nourishment where needed, repair damage, remove waste, fight infection, initiate new developments (e.g. changes at puberty), and many more.

These extremely important forms of chemically implemented intelligence on sub-microscopic scales are ignored by almost all investigations of natural or artificial intelligence. The label "natural molecular intelligence" could be used in contrast with "molecular intelligence" to refer to *human-designed* molecular biomedical tools used in nanoscale medical machinery, mentioned by Wilson(referenced above). This document is part of a project investigating varieties of "natural molecular intelligence".

Discrete and continuous processes are required

An important feature of chemistry is the combination of *discrete* quantum-mechanical mechanisms creating and removing chemical bonds, discussed by Schrödinger in 1944, and *continuous* processes of spatial change, including: changes of shape of complex molecules, changes of size and shape of brain structures, and changes of body parts during reproduction and development, as well as continuous motion while performing actions. Many such hybrid (continuous+discrete) chemical processes are involved in processes of gene expression, as well as processes in digestion, distribution of resources in a complex plant or animal body, removal of waste, repair of damaged tissue, resistance to infection, and many more.

All this suggests that purely digital, logic-based, information processing cannot meet all the requirements for production and control of forms of biological functioning and development that include continuous relative motions of complex molecules -- e.g. folding/unfolding, twisting or rotating, and also discrete changes e.g. combining physical particles to form larger structures. (Did something like this sort of thought process lead Turing to investigate chemical processes of morphogenesis?)

Could such mixed, chemistry-based, mechanisms also be involved in thought processes? Many examples of human mathematical insight and creativity in engineering design and artistic expression seem to combine continuous and discrete processing, for example thinking about use of compasses and pencil to bisect an angle in Euclidean geometry. I'll return to this later, in

discussing ancient mathematicians and intelligent animals.

One important point is clear: insofar as many mathematical discoveries are concerned with awareness of what is possible, or impossible, or necessarily the case in some configuration or class of configurations, those discoveries cannot be made by neural nets that are merely able to derive probabilities from statistical evidence. If sub-cellular chemical processes provide the required mechanisms, then a major task is to explain how they can do this: perhaps by generalising the specification of a turing machine to include not just discrete changes on a discrete tape, but operations on molecular structures, joining, separating, rotating, twisting, bringing together, and turning chemical bonds on or off. These are already operations known to be involved in processes of gene expression. [REF TRANSCRIPTION TUTORIAL]. Had Turing already guessed, by 1952, some of the important aspects of behaviour of DNA, RNA and other products during processes of reproduction and other sub-cellular processes?

Note on Euclidean geometry:

Readers who have never studied geometry and learnt to construct proofs (alas the vast majority of the "well-educated" academic population I encounter nowadays) may enjoy an excellent presentation on proving Pythagoras' theorem by Eddie Woo:

<https://www.youtube.com/watch?v=tTHhBE5lYTg>

There are many online tutorials on aspects of mathematics that are far too "mechanistic" and shallow, by comparison with Woo. Several hundred different proofs of Pythagoras' theorem have been found, some long before the time of Pythagoras. For examples see:

<https://ed.ted.com/lessons/how-many-ways-are-there-to-prove-the-pythagorean-theorem-betty-fei>

The fact that these topics are no longer a standard part of the school education, or even university mathematical education, amounts to an appalling gap in the education of future researchers, leaving many people blind to serious gaps in current AI, neuroscience, psychology and cognitive science.

Note on numbers used for counting:

Researchers studying human number competences often ignore the richness of our "natural number" system and the variety of its uses outside classrooms where children learn to answer abstract questions about numbers, without gaining any ideas about how such a system might have been developed or why it was useful. (Some gifted teachers are exceptions. E.g. [Liebeck\(1984\)](#).) Some mathematicians, logicians, and computer scientists now seem to think of numbers merely as objects that can be operated on according to rules that specify how to answer questions like

"What is $(3 + 5) * 6 - (4 - 2) * 7$?"

They don't ask: why did our pre-historic ancestors develop a system of number names, and what were they used for originally?

A possible partial history was proposed in Sloman([2016](#)), based on the practical uses of one-to-one correlations. For example, someone who goes to a river to catch fish for the family, may start by bringing the family along and fishing until each member of the family is holding one fish. After a while someone might discover (how?) that there's no need to bring the family. Instead something like a collection of small stones could be brought, after making sure that there is one stone per member of the family. Then fish can be caught until each stone has an adjacent fish. Understanding why this works, requires grasping the fact that one-to-one

correspondence is *transitive*.

Eventually (much later?) a genius might come up with the idea that instead of stones or other physical objects, the *names* of the family members could be used: repeatedly catch a fish until reciting the names while pointing in sequence at the fish shows that there is a fish for each name. Understanding this requires that one-to-one correspondence is transitive whether the sets involved contain physical objects, or more abstract entities, like names. That is, if two sets of items A and B are in one-to-one correspondence, and B is in one-to-one correspondence with set C, (and there are no overlaps between members of sets A, B, and C to cause confusion) then there *must* be a one-to-one correspondence between sets A and C whatever sorts of entities are in the sets.

Later on another genius might come up with the idea of avoiding having to use names of individuals. Instead it is useful to have a collection of standard (arbitrarily chosen) re-usable labels, like our "one", "two", "three", etc. Then before going fishing, recite those labels in their standard order while pointing at the people, and simply remember the last name. Then, at the river, as fish are caught, go through the same reciting process with all the fish after each fish has been caught, and stop when the last "fish label" is the same as the last "person label" selected earlier. (Of course this hypothetical allegory is not to be taken literally: the processes of discovery could have many variants, in different cultures, and many stages not summarised here, e.g. creating a system for generating infinitely many number names, done much better by Arabs than Romans!)

As before, understanding why this works requires understanding that one to one correspondence is transitive. So if a group of people map onto a collection of number names, and the same collection maps onto a group of fish, then the people map onto the fish. That raises the questions: *What brain mechanisms enable humans to grasp the transitivity?*, and why did it turn out when Piaget investigated children's abilities to reason about one-to-one correspondence in the 1950s, that they did not grasp the transitivity of the relation until they were five or six years old? This was unfortunately labelled "failing to grasp conservation of number", diverting attention from the mathematical structure that needs to be understood: one-to-one correspondence, now normally labelled *bijection* by logicians and mathematicians. (It is one of a collection of related mappings between two sets of objects, others being *injection* and *surjection*, as explained in https://en.wikipedia.org/wiki/Bijection,_injection_and_surjection.)

Note that "modern" answers often given by mathematicians to the question "What are the natural numbers" completely ignore the question: "What do humans use numbers for?" Instead they usually specify an abstract logical structure, which is a set satisfying logically specified conditions (e.g. Peano's axioms). For readers who are unfamiliar with that concept, there's a very good, though highly compressed, video tutorial lasting about 11 minutes, available here <https://www.youtube.com/watch?v=3gBoP8jZ1ls>

Discoveries in *applied* integer arithmetic (i.e. discoveries about uses of numbers in counting objects for practical purposes, and in reasoning about tasks based on counting) depend on understanding properties of structures and relationships that are usually ignored in work on foundations of mathematics based solely on logically formulated axioms for numbers and operations on numbers. E.g. reasoning based on Peano's axioms ignores properties of one to one correspondences between collections of non-mathematical objects. Peano's axioms don't mention the existence of anything but numbers and operations on numbers, and don't explicitly mention the possibility of counting anything, not even counting numbers!

Gottlob Frege and Bertrand Russell, following pioneering work by Georg Cantor and Richard Dedekind (See <https://mathshistory.st-andrews.ac.uk/Biographies/Cantor/>) recognised that our everyday concepts of cardinal numbers that can be used for counting collections of objects presuppose understanding that bijection (1-to-1 correspondence) must be symmetric and transitive, as explained in What brain mechanisms make that understanding of bijection possible? Piaget, around 1952, found that key properties of bijection are not understood until children are 5 or 6 years old especially transitivity: if bijections exist between sets S1 and S2, and between sets S2 and S3, then necessarily there is a bijection between S1 and S3. (Of course, he did not test them using this technical language.)

I don't think anyone knows what brain mechanisms allow discovery of what *must be* the case (necessary truths), whether that necessity concerns numbers, shapes or other objects of mathematical investigation. Piaget's last two books [Piaget\(1981,1983\)](#) report attempts to analyse understanding of possibility and necessity in children of various ages. He presents many examples, but he did not know how to design working models of the abilities used. I suspect nobody knows what brain mechanisms make it possible to understand that bijection must be transitive. (There are many publications claiming that "numeracy is innate", based on a shallow understanding by researchers of the nature of numbers and requirements for understanding them. In contrast, Piaget had read Kant, Russell and Frege and understood the problem.)

Turing does not seem to have known about Kant's distinctions, but his remarks about the difference between mathematical *ingenuity* and mathematical *intuition* when he claimed that machines were capable only of ingenuity, not intuition, seem to me to be related to Kant's distinction (mentioned above) between knowledge that is *analytic* and knowledge that is *synthetic* and necessarily true.

That distinction is discussed briefly below in the section on [Kant's three distinctions](#): the *empirical|non-empirical* distinction (an epistemological distinction), the *contingent|necessary* distinction (a metaphysical distinction) and the *analytic|synthetic* distinction (a distinction concerned with logical consequences of definitions).

Immanuel Kant's three distinctions

Kant, partly reacting against Hume's two-fold distinction mentioned above, noticed that the following three binary distinctions are different and depend on different cognitive capabilities.

1. Discoveries may be *empirical* or *non-empirical*.

Most discoveries made by perceiving and interacting with objects in the environment are empirical. In contrast, mathematical discoveries triggered by experiences, may be *non-empirical* insofar as thinking and reasoning suffices to demonstrate the truth of the discovery without using observation of the world to check all possible cases, or to sample enough to compute probabilities with high confidence.

Exactly what this ("thinking and reasoning suffices") means and how it works is very difficult to specify in the spirit that I am sure Kant intended. An example of a non-empirical discovery in this sense is "Spatial containment must be transitive". It is *possible* to discover that truth merely as a generalisation from experience (i.e. as empirical), but it is not *necessary* to do so. Moreover it is hard to see how any finite collection of examples could justify the generalisation that for all contained or containing shapes, no matter where they are, what their shapes are, how big or how

small they are, etc. containment must be transitive. For example, that would include very complex and intricate containing or contained spatial regions with shapes nobody has ever encountered or thought of.

That spatial containment is transitive can be discovered (even by fairly young children, I suspect, although it is unlikely to be innate) merely by reflecting on the nature of containment, without conducting large surveys of examples and looking for counter-examples. However, I know of nothing in current psychology or neuroscience that characterizes brain mechanisms capable of making such discoveries except as empirical generalisations. Empirical generalisations are all liable to be refuted by future examples. They cannot include *necessity* (what *must be the case*) as their content. So they don't include mathematical discoveries.

Piaget, who had studied Kant, Russell and Frege, wrote two books, published posthumously exploring some aspects of reasoning about necessity, possibility and impossibility [Piaget\(1981,1983\)](#), though I believe he lacked a deep understanding of types of information-processing mechanism capable of performing the required types of reasoning. Related work, emphasising topological aspects of spatial reasoning, is presented in [Sauvy and Sauvy\(1974\)](#), partly inspired by Piaget, but also without presenting mechanisms. Text-books of mathematics or logic include many more examples of facts that can be discovered non-empirically.

Kant claimed that many such non-empirical discoveries have two additional features, explained in **2** and **3** below.

2. Propositions may or may not be *analytically* true or false

Kant (possibly inspired by Hume's concept of truths that merely express "relations between ideas") noted that some knowledge is *analytic*, i.e. derivable purely from definitions using logic. An example (not used by Kant) is "No bachelor uncle is an only child". Anyone who understands the words "bachelor" (referring to unmarried males, usually adults), "uncle" referring to the brother (male sibling) of a mother or father, and the phrase "only child" referring to persons who have no brother or sister, can work out that if someone is a bachelor uncle then he is not married (being a bachelor) but must have a brother or sister who has at least one child, i.e. a niece or nephew. But having a brother or sister entails, by definition, not being an only child. So using nothing but logic and definitions one can establish the truth of "No bachelor uncle is an only child". Therefore it is analytic. Among other things, that implies that discovering its truth (with its normal meaning) does not require any investigation into how things are in the world. In contrast, "No bachelor uncle lives in a house made of gold" may be true, but it is empirical and therefore not analytic.

3. Propositions may be impossible, contingent or necessarily true.

Kant noted that there is another distinction that is different from both the empirical/non-empirical and the synthetic/analytic distinction, namely the contingent/necessary distinction. A proposition that is capable of being true in some possible situations and false in others is *contingent*. If it is true in all possible situations it is *necessary*. If it is false in all, then it is *impossible*, i.e. *necessarily false*. Necessity and impossibility are connected insofar as the negation of a necessarily true proposition is necessarily false and vice versa. These categories (necessity and impossibility) are often referred to as "modalities". More precisely, they are *alethic* modalities, concerned with what can or cannot exist or happen, as opposed to *deontic* modalities that are concerned with what *ought to be/should be/should not be*/the case. Saying that someone ought to do X e.g. help someone in distress (a deontic necessity) normally presupposes that it is possible not to do X, e.g. possible to refrain from helping -- an alethic possibility. So deontic necessities and possibilities depend on

alethic possibilities.

Kant pointed out that much mathematical knowledge is concerned with alethic possibility, necessity or impossibility. It is *possible* for the corners of a planar quadrilateral to lie on a circle, for example if the quadrilateral is a rectangle, and also *possible* for a planar quadrilateral to exist whose corners do not lie on any circle, e.g. a quadrilateral formed by pushing one corner of a rectangle inward towards the opposite corner, so that the quadrilateral is no longer convex. (How do you know that the resulting four corners cannot lie on a circle?) Moreover for any triangle there *necessarily* exists a circle passing through the vertices of the triangle (one of the less obvious simple theorems in Euclidean geometry).

On a planar or spherical surface containing a closed continuous line C (e.g. a circle or ellipse or an octagon) it is *impossible* for another continuous line L to exist in the same surface joining a point in the interior of C to a point outside C without any point in C coinciding with a point in L. This is just a tiny set of examples: there are infinitely many examples of different spatial possibilities, necessities and impossibilities. Many of these were discovered by ancient mathematicians several thousand years ago, long before the discovery of modern logic or the development of deductive formal systems.

Some of them can be discovered and used (perhaps unwittingly) by young children. For example if there are two boxes, B1 and B2, where B1 is inside B2, and a ball is inside B2, but nowhere to be found outside B1, then it must be inside B1. As Piaget noticed, the ability of a child to discover and use such necessary truths does not require the ability to *notice* that they are being used, or to formulate them in words. It is very likely that many of our ancestors failed to notice such truths although their brains had mechanisms able to make and use such discoveries. In contrast their brains did not have mechanisms able to make discoveries in astrophysics or sub-atomic physics, because those discoveries require prior collaborative development of an enormous amount of technology extending the capabilities of our motor systems, nervous systems and sensory mechanisms.

Although there are many known types of mutually supportive symbiotic relationships between species, human forms of collaboration go far beyond all others. It is a key feature of *human* genomes, though not all *animal* genomes, that collaborative products of ancestors can produce combinations of new physical machinery and new trajectories of learning and development which enormously extend both practical and theoretical achievements. (Compare Popper's "Third world", mentioned above.)

However, I know of no theory in psychology or neuroscience that explains what sorts of brain mechanisms make such discoveries possible. For example, psychological or neural theories that focus on learning by collecting statistical evidence and computing probabilities, are incapable of explaining how something is found to be *impossible*, or *necessarily* true. So key features of ancient mathematical consciousness, which I suspect overlap with some non-human forms of intelligence, in animals with good spatial reasoning abilities, are not explained by anything in current psychology or neuroscience. Piaget understood the problem, as shown in [Piaget\(1983\)](#), but was unable to provide explanations.

Example: slice a vertex off a convex polyhedron

For readers who have not personally had the kind of geometric discovery experience described by Kant here is an example, that, as far as I know does not occur in standard geometry text books but is closely related to examples in Shephard (1968). Try to answer this question without reading any geometry textbooks: If there is a solid *convex* polyhedron, and exactly one vertex is sliced off with a single *planar* cut, e.g. using a very thin planar saw, how will the number of vertices, edges and faces (V, E and F) of the new resulting polyhedron be related to the original three numbers? Note that this question can be given a definite answer only if the polyhedron is convex: i.e. any straight line joining two points on the surface of the polyhedron lies entirely inside the polyhedron, or on its surface.

Readers who are unfamiliar with this aspect of convexity are invited to think about lines joining pairs of points on the surface of a cube. If the two points are on the same face of the cube, the line joining them will lie entirely *on the surface*. If the points are on different faces the line joining them will lie entirely *inside the cube*, except for the two endpoints of the line. So, on a convex polyhedron any straight line joining two points on the surface of the polyhedron with either lie entirely on the surface, or will have endpoints on different surfaces while all the remaining points on the line are in the interior of the cube.

What mechanisms in your brain could enable you to make such discoveries? There are many more discoveries that can be made by thinking about solid objects. An example, with illustration, is provided here, with a question printed in blue:

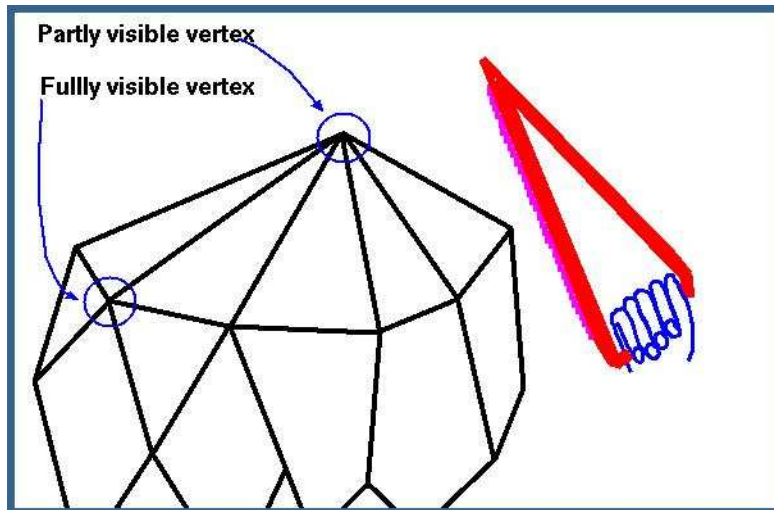


Figure Poly-Slice

Solid, opaque, polyhedron with partly visible faces, edges and vertices.

Use a planar cut that removes one vertex, e.g. the top one.

How will the numbers of vertices, edges and faces of the remaining polyhedron differ from the original numbers?

Consider [Fig Poly-Slice](#). There may be vertices, edges and faces not visible from a particular viewpoint, including edges out of sight connected to the vertex to be sliced off. You should be able to reason about how the numbers of invisible components will change, even if you don't know what the numbers are. Whether you can see the new parts will depend on how the plane of the cut relates to your line of sight. So reasoning about the numbers is not simply predicting what you will

perceive as a result of the cut.

Readers should attempt to answer the question about how the numbers will change before reading on. Try to work out an answer without reading on! This requires spatial reasoning abilities, as opposed to logical or algebraic reasoning abilities. If you find the question hard to understand, the discussion below may help to provide an understanding of the problem.

Some hints

One way to think about the problem is to imagine a 2D projected view of the polyhedron, as in the picture. Some visible vertices have all their adjoining faces and edges visible in that view, like the fully visible vertex on the left in a blue circle. That makes it fairly easy to think about what happens if such a fully visible vertex is sliced off. It is obvious (why?) where new edges and vertices will appear, and also obvious that a new face is produced by the cut. (The circles merely indicates possible sliced off vertices, and do not imply that any new circular shape will be produced.)

Some vertices are only *partly* visible from a particular viewpoint: not all of the edges and faces meeting at such vertex are visible from that viewpoint. An example is shown at the top of the polyhedron in the picture. From what is visible in that view, it is impossible to infer the number of invisible edges and faces on the far side of the polyhedron, all meeting at the selected vertex.

However, we can infer that no matter how many invisible edges and faces meet at the top vertex, a slice that removes the top vertex will produce a new convex polyhedron. In the process it will cut off a portion of each face (visible or invisible) meeting the top vertex, since the top vertex will be a sliced-off corner for each such face. The slice will also shorten the invisible edges originally meeting the top vertex, as it does for the visible edges sliced. It will also produce new edges joining the new endpoints of shortened edges.

The slice will also introduce a number of new vertices where the slice-plane intersects sliced edges in the original polyhedron and there will also be new edges connecting pairs of those vertices.

From the view of the polyhedron shown above, it is not possible to determine how many edges will be shortened and how many faces will lose a vertex. So the answer to the question will have state how the numbers change (numbers of edges, vertices, and faces), but will not be able to specify the exact old or new numbers. Before reading on, please work out your answer the question "What happens to the numbers of surfaces, vertices, and edges when one vertex is sliced off using a planar cut?"

How do you work out the answer? Do you write down (or think about) collections of statements in a logical formalism specifying the starting configuration, and then derive answers by using purely logical reasoning?

A proof is not a physical object, and need not be purely logical

In all these cases of mathematical discovery and proof using diagrams, the diagrams did not have to be physically realised: it was enough to be able to imagine them and operations on them. The ancient mathematicians who made discoveries in that way could not have used logical formalisms and techniques that were not developed until centuries later. There is no evidence that ancient human brains, and brains of other intelligent animals such as squirrels and crows, ever made (even unwitting) use of modern logical forms of reasoning, e.g. based on propositional connectives,

quantifiers, modal operators, etc.

Moreover, some of the ancient geometrical discoveries, e.g. the *neusis* construction (demonstrated in <http://www.cs.bham.ac.uk/research/projects/cogaff/misc/trisect.html>) that allows arbitrary angles to be trisected, go beyond what modern axiomatisations of Euclidean geometry can support, so it is implausible to argue that ancient mathematicians merely used logic to derive consequences from definitions and axioms specified by Euclid and formalised by Hilbert in (1899). It is not difficult to produce examples that go beyond Euclid, like the *neusis* construction..

Nobody understands the brain mechanisms?

As far as I can tell, nobody understands the brain mechanisms that make possible mathematical discoveries of the sorts Kant discussed. In addition to much standard or elementary mathematics he mentioned "everyday" discoveries such as that a left-hand glove cannot fit a right hand without first being turned inside out. (For an analysis of some unobvious complexities of this example in the light of quantum physics see Baker(2012).)

There are also discoveries (not mentioned by Kant, as far as I know) about spatial structures and processes made, and used, by pre-verbal human toddlers. Some of them are presented in a loosely organised, informal, collection of examples of "toddler theorems" [Sloman\(2011ff\)](#), e.g. "theorems" about where not to put your fingers when shutting a drawer or a door, to avoid having them squashed. Related abilities seem to be used in nest-building birds that can assemble a collection of twigs into a stable structure, e.g. crows and other corvids.

I don't believe any AI/Robotics researcher has seriously attempted to build a robot mathematician with the sophisticated spatial reasoning capabilities of toddlers or crows -- or the spatial reasoning capabilities of ancient mathematicians who made many deep discoveries in geometry, some of which seem to have been used in massive ancient engineering constructions, e.g. building pyramids, including finding, selecting and transporting suitable materials, and designing the means to achieve many subgoals, such as long distance transportation of large, heavy blocks,

Contrast this with the spatial knowledge required by termites constructing "termite cathedrals". I know of no evidence that any termite thinks in advance about the problems of transporting materials from remote sites to the required location in the nest, or considers in advance possible combinations of actions that could produce some result. Perhaps a label such as "distributed, implicit, reactive intelligence" would be an apt "high level" label for the mechanisms involved in such collective insect competences. That would also cover uses of pheromone trails during foraging. But it would not fit ancient spatial reasoning by human architects and engineers.

Why not? One reason is that there is no reason to believe that the termites, like humans, know what they are doing, or intend to do it, or can deliberately tell other termites what they are doing or teach them how to do it. Although there are ants that can (in effect) teach other ants a route to a newly found food source, as reported in 2006 <http://www.bristol.ac.uk/news/2006/879.html>, there is no evidence that the teaching ants know they are teaching or that the learners know they are learning. This is also true of much transfer of information between human parents and their offspring. When humans know what they are doing they can think about how they do it, inform others about how to do it, and sometimes notice, or seek, opportunities to use better ways to do it. These are all evidence of self-reflective capabilities in human brains. (I am not claiming that such

capabilities always lead to truth: self-deception is not unusual.)

Information-based control

An important aspect of biological evolution is repeated development of new information-based control mechanisms, using new kinds of information based on new ontologies -- metaphysical extensions.

This comment applies even to very low-level control mechanisms, such as mechanisms involved in production of new cells, or control of behaviour of cells, including control of direction of growth of cells in different parts of plants, or control of muscle contractions in animals or control of secretion of digestive juices. Control can occur in different locations, for different purposes, on different scales, and using different control structures. Simple control processes may select from a fixed set of options. Much more sophisticated control processes may have to create new options before selecting (e.g. in planning mechanisms).

For example in many plants there is a base with a fixed location in soil from which grow other parts in different directions with different functions that do not need to be closely coordinated (e.g. growth of new leaves and extension of root systems). In those cases control can be mostly distributed and localised with many subsystems sensing and acting on their immediate environment, such as roots providing nutrition and support in soil, while other parts grow stems or trunks, protective coverings (e.g. bark) branches, leaves, petals, reproductive parts, or transfer chemicals in various directions, either as nourishment, or as control signals.

In contrast, for many animals there are similar problems of control of growth, and distribution of chemicals and information between subsystems, including information about states and requirements of many body parts (e.g. growth of new materials, repair of damage, or fighting local infections, energy requirements), and in addition major requirements for obtaining information of various kinds about the environment and its occupants and globally coordinating actions of many body parts (including sensors) in order to gain information about and to interact with portions of the environment, including remote contents.

XXXXXXX

For example, an animal jumping over an obstacle while running needs to co-ordinate visual and other information about itself and its environment (including sensing the current configuration of body parts and their relative motions, the type of surface underfoot, current speed and direction of motion of the whole animal, obstacles being approached and information about whether they can be by-passed, jumped over, climbed over, etc., motion of other animals (predators, prey, competitors), locations and motions of various body parts, gaze direction and fixation points of visual sensors. This requires coordination of many subsystems developed over millions of years of evolution, where much of the control has to be delegated to expert sub-systems (e.g. in the cerebellum) That requires new controlling mechanisms to develop new forms of physically expressed *symbolism* for use in sensors, reasoners, decision makers and action controllers. A side effect of this is development of new ontologies for entities, their states, properties and relationships, and their forms of interaction. In that sense evolution of intelligent systems depends on evolution's metaphysical creativity. More details are proposed later.

Below, I'll develop a (still imprecise) conjecture that the kinds of human spatial reasoning discussed by Kant, are partly related to abilities in other species with high spatial intelligence, such as squirrels and crows, and depend on complex sub-neural chemical mechanisms that have not yet been identified.

Despite the current fashion for use of neural nets, statistics-based mechanisms, including neural nets, are totally incapable of discovering, and using, or even representing kinds of mathematical/spatial *impossibility* or *necessity* that have roles in intelligent planning prior to acting, and are part of the content of ancient mathematical spatial knowledge, for example the sum of lengths of any two sides of a planar triangle must always exceed the length of the third side. Why?

Kant's distinctions between kinds of knowledge seem to be relevant to some of Turing's ideas, although, as far as I can tell, Turing had never read Kant and did not explicitly make use of Kant's distinctions. However he (Turing) made related claims about mathematical knowledge, in particular when he wrote, in his thesis, that there is a distinction between mathematical ingenuity and mathematical intuition, claiming that only the former (ingenuity) could be achieved by computers, although human mathematicians are also able to use mathematical *intuition*.

As far as I know, Turing never explained what he meant by the distinction between ingenuity and intuition (especially what he meant by "intuition"), so I may be over-interpreting his words when I suggest that his distinction is related to some of Kant's claims, which, in turn, seem to be related to important developments in biological evolution that have largely gone unnoticed. I'll try to show that there are connections with points made by Schrödinger in his book *What is life?* (1944). Schrödinger drew attention to important, but largely unnoticed, relations between quantum mechanisms and biological evolution -- relations that later turned out to have deep significance for a wide range of mechanisms and processes in evolution and development. I'll try to explain that this is relevant to evolution of mathematical capabilities, in the section below on [Chemical Information Processing](#).

It is hard to understand how biological evolution could have produced brains with mechanisms able, thousands of years later, to acquire many kinds of mathematical knowledge, including both highly complex, highly abstract, types of modern mathematics and also ancient topological and geometric knowledge concerning possible and impossible structures and processes -- i.e. not mere statistical knowledge of relative frequencies.

Between the earliest, pre-historic, organisms with no such competences and humans who were able to make mathematical discoveries, there must have been many intermediate cases in which different but related capabilities evolved -- but not a *continuum* of cases, since there can be only a finite number of intermediate individuals between any two members of a thread of biological inheritance. In any case, the evolution of physical machines composed of *discrete* physical particles cannot have a continuous trajectory. It follows that a full explanation of the evolution of any complex biological capability must be made up of many *discrete* steps that together produce new mechanisms and capabilities.

However, much of the intermediate evolutionary learning can produce information stored in the environment (e.g. in products of human intelligence, including new shared languages and technological abilities) rather than in the genome. (This is related Karl Popper's "Third World". See (1972)) For example, even when the first individuals with a "fully fledged" human genome emerged, none of the individual humans could have made mathematical discoveries that were made by their

descendants much later, just as none of them could have created a language like English, French or Urdu, or thought of Newton's laws.

Relevance to Turing

Perhaps an interest in finding possible evolutionary trajectories leading up to current human brain functions led to Turing's work on chemistry-based reaction/diffusion mechanisms producing varied surface patterns reported in [Turing\(1952\)](#). However, those could not have been the mechanisms he had in mind when he wrote, in [\(1950\)](#), "*In the nervous system chemical phenomena are at least as important as electrical*", without explaining why! Later I'll return to problems of using chemical machinery to construct new chemical machinery in accordance with evolved construction plans, and how that might have contributed to precursors of mathematical intuition.

I suspect Turing's work on production of surface patterns was a temporary diversion followed partly because it was interesting in its own right and indirectly relevant to his main problem, and partly because he was finding his main problem (specifying mechanisms of mathematical intuition) too difficult at that time! (He explicitly denied that intuition could be replicated on computers, although ingenuity could.)

In the rest of this paper I'll sketch some ideas about what Turing might have noticed about mathematical discovery, and relate them to much older ideas that come from Immanuel Kant, mentioned above. The main result will not be to finally answer questions that interested both of them, but to suggest some small steps towards understanding evolution, development, and functioning of brain mechanisms related to ancient mathematical discoveries and also related to many other biological phenomena involving varieties of information processing, especially processing of *spatial* information, and its use in spatial control.

This will draw attention to aspects of human mathematical abilities and forms of mathematical consciousness that tend to go unnoticed, including the many different developmental stages required: I suggest there are far more evolutionary and developmental stages than psychologists have noticed, including changes in sub-neural chemical information processing mechanisms cannot be observed in standard laboratory experiments.

Evolved "intermediate-level" control functions and mechanisms

Fine-grained control of intricate physical processes cannot be based on information about the interactions and trajectories of all the sub-atomic particles involved, since all that detailed information is not available to decision-making control mechanisms in brains, and even if it were available, processing it all in real time would be an intractable task, not least because the mechanism doing the controlling would then have to internalise all the physical complexity of the system being controlled. How could the processes in the controller be controlled, without an infinite regress? A possible way out is to aim to control processes at a level of abstraction that omits all or most of the fundamental physical detail, and uses only information about macro features of structures and processes, just as human engineers, design, build and control machines at a "macro" level that ignores a vast amount of sub-microscopic physical detail. I'll expand this idea and some of its implications [below](#), relating it to aspects of design and control of human-made machines.

If this is correct then evolution implicitly produced and used multi-layered metaphysical concepts and theories long before humans did!

This suggests (or implies!) that evolution of increasingly complex biological control mechanisms requires evolution of increasingly complex mechanisms for discovering and using increasingly abstract information structures for real-time/online control. For that to occur biological control mechanisms produced by evolution would need to be able to discover and use information about key *subsets* of the relevant physical reality, just as human designers and controllers of complex machinery have to ignore most of the sub-atomic quantum mechanical details (or details of nuts, bolts and other fragments of a complex design) and find ways of identifying and using, with the aid of appropriate sensors and effectors, a tiny subset of (relatively) high level control parameters that suffice for successful online, real-time, decision making by specialised control and reasoning mechanisms. (I'll illustrate that below with the problem of controlling a dockside crane.)

As evolution produces larger and more complex organisms, or controllable parts of organisms, this will require repeated introduction of new information levels used by new control mechanisms for real-time control of increasingly complex bodies, or sub-systems, interacting with increasingly complex environments (eventually including other organisms).

Later still, evolutionary developments in some species, including humans, extended those mechanisms to include collaborative and non-collaborative interactions with other intelligent individuals, including systems of teaching/educating and collaborative design and construction of new structures, procedures and theories, and internal and external tools to facilitate such processes. Evolution of mechanisms like camouflage, or abilities to deceive other organisms, depends on the facts about abilities of other organisms to take in and use information about structures and processes to take control decisions. Trees don't do that, so you can't *fool* a tree, although control of lighting, humidity and other features of the environment can be used to *influence the behaviour of trees* in botanical gardens.

In humans, later stages of those evolutionary processes extended the information contents of processes of learning, goal formation, planning and decision making, to include more of the environment, past, present and future, eventually including theories about information processing capabilities of and information available to other individuals, including prey, predators, collaborators, and their own offspring.

The resulting new forms of collaboration and communication *between* individuals, also included evolution of mechanisms that create and extend the volume and spread of information stored *within* individuals, and used for interacting

Alan Turing and Immanuel Kant

Does ancient spatial reasoning need quantum mechanisms? Although Turing doesn't mention Kant, he (Turing) held a view of mathematical discovery that seems to me to have been close to Kant's view, at least as I understand Kant's view, summarised earlier. As mentioned above, Turing's PhD thesis, published in [\(1938\)](#), mentions a distinction between mathematical ingenuity and mathematical intuition, claiming that only the former (ingenuity) could be achieved by computers (e.g. Turing machines). As far as I know he never explained clearly what he meant by the distinction between ingenuity and intuition, and why he thought computers were incapable of intuition, so I may be over-interpreting his words in what follows.

One possible interpretation is that what Turing meant by the use of *ingenuity* includes only the use of definitions and logical inference steps in discovering what Kant called "analytic" truths, all of which are logical consequences of definitions, as explained below. (An example is "no bachelor uncle is an only child". How would you convince yourself that that must be true?)

I suspect (but cannot demonstrate) that Turing's interest in mathematical *intuition*, gaining insight that goes beyond analytic truths, to discovery of *synthetic* but non-empirical knowledge, led him to consider chemical information processing, mentioned obliquely in his [1950](#) remark about the importance of chemistry in brains. He did not explain why it was important.

I'll offer reasons for that claim, which may or may not have been Turing's reasons! He was an active member of the Ratio Club, whose history is presented in *Husbands and Holland* ([2008](#)). Unfortunately that gives no indication of Turing's reasons for referring to brain chemistry, though it reports that he gave a talk on "The Chemical Origin of Biological Form", presumably summarising his 1952 paper on morphogenesis, ([1952](#)), on which I'll say more below.

Since Turing's death, much has been learnt about gene expression and brain chemistry that he could not have anticipated in detail. But he may have anticipated some general features of chemical processing in brains, summarised below, that he never spelled out in his publications. If he had spotted those features, that could have provided the basis for his remark (in 1950) about the importance of chemistry in brains. Perhaps it was not mentioned in his 1952 publication explicitly related to chemistry for the simple reason that the two topics (external pattern formation, and mathematical intuition) are unrelated even if both depend on chemical processes. The 1952 paper focused only on processes producing *patterns on surfaces* (e.g. *skins and fur*) of animals or plants, without mentioning *reasoning mechanisms in brains*, or processes in organisms without brains that acquire and use information. Perhaps the detailed additional *internal* roles for chemistry were not mentioned because Turing was still developing his ideas about chemical information processing in brains, tantalisingly mentioned in his 1950 paper. Or perhaps I have over-interpreted what he wrote about chemistry!

Below I'll summarise some implications of the discovery of the structure of DNA, and later work on gene-expression using DNA, RNA and derived chemical structures. This helps to show what chemical mechanisms can add to forms of sub-cellular computation in control of biological mechanisms of varying levels of complexity. Not all the mechanisms need to be directly specified in the genome: some are results of interactions with the environment at different stages of development -- interactions that vary enormously across types of organism, and in some cases also between individual members of a species.

Chemical Information Processing

Chemical processes include both continuous and discrete processes. Continuous processes include molecules, or parts of molecules, altering their spatial relations during rotations, translations, twisting and various combinations of those processes. Discrete chemical changes include formation or alteration of bonds between particles in molecular structures. The possibility of such bonds is explained/predicted in quantum physics, but neither Newtonian physics nor Einstein's General Theory of Relativity says anything about chemical bonds.

The importance of quantum physics for biological mechanisms involving bonds between particles was stressed in Schrödinger's *What is life* ([1944](#)). (The book was published several years before

the work of Watson, Crick and Franklin, on the structure of DNA, work that was in part inspired by Schrödinger's book.) In that book Schrödinger pointed out that chemical bonds made possible by quantum mechanisms are capable (in principle) of explaining (or helping to explain) *reliable* biological reproduction, a topic on which Newtonian physics is helpless.

Newtonian physics cannot even explain *existence* and persistence of chemically complex parts of animals, plants and other organisms, let alone explain their reproduction. In particular, Newtonian particles cannot form bonds. So Newton could not explain the existence of rocks -- whose solidity and rigidity depend on chemical bonds -- let alone properties of bones, skin, sinews, nervous tissue, tree-trunks, leaves, petals, seeds, and other parts of organisms.

Whether this was intended or not, Schrödinger's remarks about differences between Newtonian and Quantum physics are relevant not only to explanations of reliable reproduction, but also to requirements for many forms of information processing and control in organisms, during development from a fertilised egg, seed or spore, growth by extension and reproduction of existing parts, and performance of actions of many kinds, including building nests, caring for young, escaping from prey, or developing new shoots after being blown from a parent plant to a new location. All such control processes depend on chemical interactions in brains, which, in turn, depend on quantum mechanisms.

Having made such points, on p. 171 of his [\(1967\)](#) Schrödinger draws a pessimistic conclusion regarding how much can be explained:

We can follow the pressure changes in the air as they produce vibrations of the ear-drum, we can see how its motion is transferred by a chain of tiny bones to another membrane, and eventually to parts of the membrane inside the cochlea, composed of fibres of varying length, described above. We may reach an understanding of how such a vibrating fibre sets up an electrical and chemical process of conduction in the nervous fibre with which it is in touch. We may follow this conduction to the cerebral cortex and we may even obtain some objective knowledge of some of the things that happen there. But nowhere shall we hit on this 'registering as sound', which simply is not contained in our scientific picture, but is only in the mind of the person whose ear and brain we are speaking of. We could discuss in similar manner the sensations of touch, of hot and cold, of smell and of taste.

An answer to this concern is provided by the development and use since the 1970s of increasingly powerful types of *virtual machinery* that can be implemented in changing collections of physical machinery, including self-monitoring and self-modifying virtual machines, many of which now run on the Internet. I expect none of this would have been surprising to Turing, who had previously shown how versions of his machine could, in principle, go through the same processes as any other machine in a wide class of machines (hence the "universal" in "Universal Turing Machine"). It is arguable that biological evolution "discovered" many of the powers and uses of virtual machinery long before humans did! For more on this topic see the Appendix: [Varieties Of Virtual Machinery below](#). However, so far no human-designed virtual machine has come close to the power and sophistication of virtual machines produced by biological evolution (extended by social and cultural evolution), such as human minds. Even the minds of many other intelligent species (e.g. squirrels and magpies) are unmatched, as regards their expertise, by any current AI system.

The inadequacy of Newtonian mechanisms

Returning to Turing's interests, the mechanisms involved in ancient types of mathematical reasoning about spatial structures and processes, e.g. in Euclidean geometry, could not be implemented in particles whose behaviour is controlled only by Newtonian influences, such as gravitational attraction and bouncing off other particles. Processes in a weight-driven clock controlled by a pendulum can be given a Newtonian explanation, but the existence of the materials making up the clock cannot.

Which features of quantum physics make mathematical reasoning possible in brains is far from clear, but the existence of complex molecules and the formation and removal of chemical bonds, and effects of electrical charges are all potentially relevant (as indicated by Schrödinger). More generally, insofar as many ancient mathematical discoveries were concerned with possible and impossible spatial configurations, and consequences of changes in those configurations, the biological mechanisms in brains concerned with space must be able to represent those physical states and processes, and the properties possessed by various combinations of physical objects.

Mathematical reasoning about spatial structures and processes in humans seems to be an extension of types of spatial reasoning in a wide variety of non-human animals. (This obviously does not include the kinds of spatial reasoning used by an octopus!) A difference between humans and (most? all?) other animals is that humans can reflect on and explicitly think about and teach others what they have learnt, using linguistic and other forms of thought and communication, whereas other animals that have spatial competences lack such powerful meta-competences. Moreover humans are able to discover that some describable states and processes are **impossible** and that there are states that are **necessary** consequences of other states: for example, that object A is in container C is a necessary consequence of A being in container B, and B being in C.

Building on fairly recent research on processes of molecular transcription, I'll try to show, that very primitive meta-competences making use of spatial information might have been used for control of sub-cellular spatial assembly processes from early stages of evolution, and in early stages of individual development and also in the later stages of physiological control, of many complex organisms.

Many of those processes involve machinery that reads and transcribes molecular sequences (e.g. in DNA and RNA) while assembling new molecules for a variety of purposes. In some cases the main purpose is simply making a new copy of genetic information, e.g. in cell division. Other cases include using information in a pre-existing molecule to control construction or assembly or form new physiological materials or components. In others the main purpose may be processing of information, e.g. deriving specifications for later assembly processes or other control processes.

Some of these processes can be thought of as loosely analogous to online assembly of branching new machines, that continue to operate in parallel. Later versions of such processes, may operate within brain cells, performing operations on information structures to create new information structures, rather than growing new body parts. I'll return to implications of this idea below. If Turing had had similar ideas by 1950 that might have justified his mysterious comment about the importance of chemistry in brains, but I expect we'll never know whether he had. He was writing before the flood of research on types of chemical transcription and their uses in organisms.

Evolution and development of spatial reasoning

Euclid's *Elements* and many records of ancient discoveries of geometric constructions, possibilities, impossibilities, necessary connections, and proofs, demonstrate that ancient thinkers had abilities to reason about effects of non-verbal spatial constructions, and the structural relations between resulting spatial structures, using thought processes that were very different from construction of sentences in reasoning.

Those ancient forms of spatial reasoning did not require use of perfect physical devices (straight edge, and compasses), only abilities to *think about* perfect versions, in order to discover spatial necessary connections and impossibilities, e.g. discovering that it is not possible for two planar triangles to have corresponding sides with equal lengths, while the corresponding angles in the two triangles are not the same size. I.e. fixing lengths of sides fixes sizes of angles, in a triangle. (Think about why that isn't true if there are more than three sides.)

Such thinking used brain mechanisms whose abilities to discover spatial possibilities, impossibilities and necessities, depend on mechanisms of spatial intuition that are not understood by neuroscientists -- or anyone else -- at present, as far as I can tell.

Moreover, there are closely related reasoning abilities in other intelligent animals, and in pre-verbal human toddlers (mentioned above), shown by choices of actions, about which those agents are unable to communicate verbally, or to reason about using a human language. Squirrel intelligence, using similar abilities, can be contrasted with snail/slug intelligence, in both cases used to get at nuts in a bird feeder, as discussed here, with remarks about some differences between squirrel and slug intelligence:

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/squirrel-intelligence.html>

Some readers may be aware from television programmes and books like Godfrey-Smith (2017) that octopus brains are able to support staggering kinds of spatial intelligence.

Immanuel Kant was familiar with the kinds of human spatial reasoning capabilities involved in ancient mathematical discoveries, as they used to be a standard part of mathematical education (also in my youth), though now (in 2020) they are no longer taught in most schools, and are ignored by many researchers trying to design computer models capable of explaining or replicating human forms of learning and discovery including discovering mathematical necessities and impossibilities (emphasised by Kant). As a result of this educational gap, most modern AI researchers focus only on uses of statistical evidence and derived probabilities, using tools and strategies that are *incapable* of explaining some of the most important varieties of mathematical cognition, since ancient times.

There are also AI researchers producing models of logic-based intelligence plus algebraic manipulation, which cannot explain ancient forms of mathematical discovery made long before the development of modern logic and algebra. However, those techniques, unlike neural nets, are able to establish (non-empirically) that some propositions are necessarily true and some are necessarily false.

So two important research communities both ignore the ancient forms of mathematical reasoning referred to by Kant, which seem to be important not merely in mathematical discovery but also in intelligent control of actions in space. Ignoring them is likely to lead to robots that cannot be trusted in spatially complex tasks, like putting a nappy (diaper) held by safety pins onto a wriggling baby.

Note: Software tools such as the amazing freely available Geogebra system that is used to animate famous results in geometry, e.g. Morley's theorem, activated here: <https://www.cut-the-knot.org/triangle/Morley/>, are not proposed as models of human spatial reasoning, only as aids to such reasoning in humans. They also indicate some of the sophistication of spatial reasoning in ancient mathematicians. A list of demonstrations available is here: <https://www.cut-the-knot.org/geometry.shtml>

My informal enquiries suggest that most learners (in the UK and many other countries) now do not encounter Euclidean geometry, and the kinds of spatial reasoning using diagrams (imagined or created on visible surfaces) that were taught to bright children in school mathematics classes until about the middle of the 20th Century leaving a huge gap in their education. The ancient spatial reasoning capabilities required by the ancient geometers are not yet replicated in AI, or explained by theories in psychology or neuroscience. Nearly all researchers in those fields ignore the combination of facts pointed out by Kant, including the fact that many ancient mathematical discoveries were concerned with what is *impossible* or *necessarily true* -- features that have nothing to do with probabilities derived from statistical evidence.

So recent AI theories and mechanisms based on artificial neural nets collecting *statistical* evidence and computing *probabilities* are not up to the task of replicating ancient mathematical competences -- They are completely irrelevant to that task! However, they may be useful aids to researchers, e.g. when used to explore large spaces looking for potentially useful regularities. (Thanks to Jon Rowe for pointing that out to me.)

Logic-based computation

The purely logic-based forms of computation starting from Hilbert's axiomatisation of Euclid [Hilbert\(1899\)](#) also do not replicate or explain ancient forms of spatial reasoning about geometric and topological necessity or impossibility. However, although the ancient discoveries about Euclidean space are not derivable from definitions using pure logic, some superficially similar conclusions are, e.g. theorems derivable, using only logic, from Hilbert's (or some other) logical axiomatisation of Euclid. However, Hilbert's proofs provide information about spatial structures only if combined with a proof that his formal system accurately characterises space as *experienced* by Euclid and his predecessors (over several centuries) whose discoveries Euclid systematised.

Of course more sophisticated observations by physicists and astronomers in the 19th and 20th centuries revealed that locally experienced Euclidean portions of space are embedded in a much larger space that is non-Euclidean. That did not imply that engineers, architects and many others should all abandon Euclidean geometry, on which so much science and technology had been built successfully. And it does not deal with the problem of identifying the cognitive mechanisms making that ancient reasoning possible, and replicating those mechanisms in human-designed reasoning machines, if possible.

I shall try to point out features that are missing from current models of mathematical discovery, and propose a research programme that may lead us to the required mechanisms. I suspect, but cannot prove, that Turing was trying to do something similar. His [1952](#) paper included discoveries made as a *side effect* of that investigation.

My suspicion is based on the mismatch between the contents of Turing's morphogenesis paper and the implications of his obscure remark about the importance of brain chemistry in his 1950 paper. That remark could not have been based on thinking about formation of external patterns

made of dots, blotches, spirals, etc. or similar patterns in brains. What was he thinking of? We'll never know, but the latest available writings and discussions by Turing, assembled in Copeland's collection [The Essential Turing \(2004\)](#), do not support my hunch. So most of this document is a wildly speculative extension of Turing's ideas rather than a report and discussion of his ideas.

I suggest that the mechanisms referred to by Kant, and used by ancient mathematicians, depend on the conjectured sub-neural chemical (molecular) information-processing, about which Kant knew nothing, and we still don't know enough!

All forms of biological reproduction and development depend on such molecular machinery, but there is not a unique type of machine for reproduction and development -- analogous to the role of a Turing machine in theoretical computer science. Instead, in humans and other intelligent animals, there are many different chemistry-based machines that evolved at different times and, and an even greater variety of chemistry-based forms of construction and control in biological organisms of many kinds. A partial survey including historical speculations is offered in Sloman([2020](#)), but the ideas are still being developed, as explained below.

Relevance to Turing

Is all this relevant to Turing? The goal of explaining or modelling ancient spatial reasoning was not mentioned as a motive for studying chemical mechanisms in Turing's 1952 paper on chemistry-based morphogenesis [Turing\(1952\)](#). Turing's sentence about chemical phenomena being important in brains, quoted above suggests that Turing thought there was a connection, though he did not give that reason.

I suspect Turing did not know as much about brains as his contemporary Kenneth Craik who also died tragically young, in 1945, after publishing Craik([1943](#)) including potentially relevant deep remarks in his discussion of mechanisms of perception, e.g. asking how a tangled network of neurons can represent straightness, though he is better known for the claim, later in the book, that brains build "models" of what they perceive in the environment, and use those models in planning and controlling actions. I have not been able to find out whether Turing and Craik, two of the deepest thinkers of the time, ever met, or knew of each others' work. Perhaps the combination of Craik's deep knowledge of brain physiology (leading him to wonder how a tangled mess of neurons could detect or represent straightness, for example) and Turing's deep mathematical and computational insights might have produced important discoveries that have not yet been made seven decades later. It is possible, though I know of no evidence supporting this, that Craik had considered the possibility that sub-neural molecular mechanisms are crucial to ancient kinds of mathematical reasoning, though this was before the discovery of the structure of DNA. Perhaps someone reading this paper will report something relevant in Craik's work.

There is a deep but less obvious connection with themes in Schrödinger's extremely influential little book *What is life?*([1944](#)), though I have no evidence that Turing had read it when writing about chemistry a few years later. I also know of no evidence that Schrödinger thought chemical mechanisms in brains might be relevant to ancient mathematical discovery processes. Nevertheless, it is possible that the mixture of *discrete* chemical changes (e.g. chemical bond formation or modification) discussed by Schrödinger, along with *continuous* spatial deformation that can occur to molecules suspended in a chemical soup, as discussed by Turing, will turn out to be relevant to the kinds of mathematical intuition supported by human brains, of which Turing thought computers were incapable.

A very interesting discussion, linking mathematics, Kant, Turing, biology, evolution, development, and genetic codes, is in [Roth\(2011\)](#) (discovered when I had nearly finished this paper).

Turing's best known work on chemistry ([1952](#)), showing how reaction/diffusion processes can form patterns on external surfaces of organisms appears to be completely irrelevant to mathematical intuition and spatial reasoning. But perhaps he regarded that work as an interesting preliminary investigation, to be followed later by a shift of focus from "macro" movements of continuous fluids close to the surface of an organism (forming skin patterns) to sub-microscopic molecular interactions within brain cells, involving both continuous processes (e.g. items folding, twisting, or moving together or apart) and discrete switching (e.g. forming and releasing chemical bonds). Those combinations are not possible in Newtonian physics. That was the basis for Schrödinger's claim in ([1944](#)) that quantum mechanisms (including formation and switching of sub-molecular bonds) are essential for life. But he was thinking mainly about processes of reproduction, not reasoning.

We now know that those processes of reproduction include chemical machinery for assembling chemical machinery, including mechanisms that assemble molecules in a chemical soup to form strands of RNA while they "read" specifications in strands of DNA. Later, strands of RNA are read to assemble other molecules. Some of the mechanisms are described and depicted in Hoffman's [video lecture](#) on sub-cellular chemical machinery concerned with processes of reproduction. As chemical control mechanisms became more complex, because they were controlling more complex mechanisms, including for example construction of materials required for cell walls, bones, nerves, muscles, sinews, skin, digestive juices, blood, sap, (and many more), the chemical mechanisms must have used evolved forms of information processing referring to important features of *processes being controlled*, rather than simply to the "bottom-level" physical particles or waves involved.

For example, during digestion in an animal, portions of consumed animal or plant matter need to be detected and disassembled into re-usable smaller components some of which can be transferred to other parts of the body for use in construction or repair or supply of energy, some discarded as waste, to be transported to waste outlets, some assembled into antibodies required for attacking/disabling invasive materials or organisms, some used for constructing reproductive materials to be stored for later use, or passed to other individuals in the case of sexual reproduction, and many more. These processes, and the materials used, will vary across organisms, and across parts of the body in a single complex organism, all controlled using molecular information. xxxx GANTI

Although there is much still to be learnt about the earliest life forms, Tibor Ganti, using knowledge available by the 1970s, produced a specification for the simplest possible self-sustaining, self-reproducing life form, a single cell, the "Chemoton", that could survive and reproduce in a suitable chemical soup. The ideas are usefully summarised and reviewed in Korthof ([2003](#)). Such an organism requires a surprisingly complex collection of different components, all of which need to be maintained during normal life and duplicated during reproduction. Whatever the simplest life forms were, evolution thereafter repeatedly extended the ontologies and forms of representation used by newly evolved control mechanisms -- long before human theorists began to produce new scientific ontologies. A key point is that even in the simplest such organisms information of different kinds is constantly being used in different parts of the organism to control what happens. Those processes will depend crucially on the features of quantum physics discussed by Schrödinger. So unlike a Turing machine, in which only bit patterns are read, modified, copied, and stored on the

tape, organisms constantly assemble, disassemble, copy, store, transmit, molecular structures, and in the process produce new components doing more of the same, but with variations needed in different parts of the organism. Could it be the case that related mechanisms play a role in sub-cellular processes of control and reasoning, generalising the notion of a Turing machine in several ways, including repeatedly producing new branching structures instead of merely modifying a single linear structure.

A useful online tutorial on molecular machinery relevant to reproduction and genetically controlled development of organisms is available in Hoffman's [2012 video lecture](#). There are many more recent online lectures and tutorials including spectacular synthetic videos illustrating molecular processes, such as DNA or RNA transcription. (I hope someone will produce an easily accessible online tutorial presentation of the variety of such mechanisms and their biological uses: including the changing control ontologies required as organisms become more complex.)

In a purely Newtonian universe -- with no chemical bonds or bond-changing reactions -- life, and even rocks, would be impossible. A full account of how evolution produced the mechanisms making possible ancient mathematical minds, and what those mechanisms are, will include as yet unknown exceedingly complex processes of evolution and individual development that make use of currently unknown sub-neural chemical mechanisms. As evolved organisms became more complex, the forms of control became more complex and varied and increasingly dependent on new forms of *virtual machinery* implemented in *chemical machinery*. The relationship of such new virtual machines to molecular substrates are far more complex and varied than relations of human designed virtual machines to current digital electronic substrates, including internet-based virtual machines distributed over complex, constantly changing physical machinery, interacting with complex and changing environments of many kinds. (See [the Appendix](#) below.)

Note: Penrose on sub-neural mechanisms

The physicist, Roger Penrose has argued in favour of sub-neural mechanisms to explain ancient geometrical reasoning, e.g. in this September 2019 video at a conference on models of consciousness in Oxford: <https://www.youtube.com/watch?v=3trGA68zapw>. However, as far as I can tell, he has not yet specified sub-neural mechanisms with appropriate computational powers for detecting geometric necessity or impossibility, though the lecture ends with reference to "an element of proto-consciousness", which does not seem to me to provide an explanatory mechanism. (My own talk at the same conference attempts to state the problem of accounting for consciousness of geometric/topological impossibility or necessity, but does not claim to provide any mechanism.)

[TO BE RE-WRITTEN OR REMOVED]

Biological Mechanisms of Mathematical Discovery

The Meta-Morphogenesis project

Until fairly recently (the first half of the 20th Century), Euclidean geometry was a standard part of a good mathematical education -- including finding geometric proofs or counter-examples, finding geometric constructions, proving properties or limitations of such constructions, etc. That gave students first-hand experience of replicating some of the achievements of great mathematicians of the past, including independently making discoveries presented in Euclid's *Elements*. In some cases they made discoveries that went beyond Euclid, for example the (re-)discovery, in the early 1970s, by Mary Pardoe (a young mathematics teacher) of a proof of the triangle sum theorem, using a construction not included in Euclid, based on repeated rotation and translation of a line

segment, and not derivable from Euclid's constructions and theorems, as explained here: <http://www.cs.bham.ac.uk/research/projects/cogaff/misc/triangle-sum.html>

Solving the sliced polyhedron problem presented [below](#) requires geometric reasoning processes that are not part of Euclid's system.

Hilbert's "logicisation" of Euclidean geometry [Hilbert\(1899\)](#) is often used to claim that ancient discoveries in geometry, reported in Euclid's *Elements*, are all based on purely logical derivations from a set of axioms presented by Euclid. However this does not explain how the axioms and constructions in Euclid were originally discovered, long before the use of formal logic-based proofs.

Moreover there were constructions and proofs known to ancient mathematicians, and some discovered more recently, that are not derivable from Euclid's axioms. So the combination of Hilbert's presentation of Euclid's axioms along with mechanisms of logical deduction cannot explain all ancient mathematical discoveries regarding spatial structures and processes.

For example, ancient mathematicians (e.g. Archimedes) knew of and made use of geometric constructions that went beyond the power of Euclidean geometry, but for centuries those constructions were disparaged by mathematics teachers and not mentioned in textbooks on geometry, until very recently. An example is the physically realisable *neusis* construction that makes it easy to trisect an arbitrary angle, which is not possible using only Euclid's constructions. The construction is described and discussed further in <http://www.cs.bham.ac.uk/research/projects/cogaff/misc/trisect.html>

Such examples refute claims that Hilbert had shown that all geometrical reasoning is implicitly logical reasoning, a claim that is often used against Kant, e.g. [Hempel\(1945\)](#).

As a result of reflecting on experiences in which necessary connections or impossibilities involving spatial structures and processes are discovered, Immanuel Kant [\(1781\)](#) claimed that there are kinds of mathematical discovery that produce knowledge with three features that had not previously been clearly distinguished, although David Hume came close with his division of kinds of knowledge into two categories, roughly: *empirical*, labelled "Matters of fact" by Hume, and *conceptual*, labelled "Relations between ideas" by Hume. (This is a shallow summary, not to be confused with deep Hume-scholarship!)

Kant was provoked into criticising Hume's two-fold account by making three different binary distinctions, explained below. I suspect that Alan Turing's investigation of chemistry-based morphogenesis was in part motivated by a belief (or hunch) that chemistry-based brain mechanisms could play a role in mathematical reasoning that cannot be replicated in digital computers. Whether he had that hunch or not, I think it was correct. But I also think the reaction/diffusion mechanisms discussed in [Turing \(1952\)](#) are the wrong sorts of chemistry-based mechanisms for explaining mathematical reasoning, though it is likely that Turing knew that!

Someone with the imagination and flexibility of Turing would have been enthralled by the ideas of molecular assembly machinery discussed and demonstrated recently e.g. by Hoffman, as mentioned above, and would have begun to explore ways of generalising Turing machines, by replacing them with machines that, as they traverse one discrete linear structure, repeatedly create derived new structures that can later be assembled in various ways to produce new materials and structures, including new and more complex "derivation machines", using a mixture of discrete

changes (e.g. catalytic switching of molecular bonds) and continuous changes (e.g. folding, twisting, coming together, moving apart, along with mechanisms of spatial control). I suspect some human forms of mathematical reasoning about spatial structures and processes (by mathematicians, engineers, architects, dress-designers, and many more) make essential use of such chemistry-based sub-neural information processing, justifying some of Kant's claims about mathematical discovery.

I have not found any evidence that Alan Turing was acquainted with, or approved of Kant's work. However, in 1936 he mentioned a distinction between mathematical *ingenuity* and mathematical *intuition*, neither of which he attempted to define precisely (as far as I know). He also suggested that human mathematicians have both ingenuity and intuition, whereas computers can only have mathematical ingenuity. Unfortunately, he did not explain precisely what he thought the differences were between mathematical ingenuity and intuition, and why he thought only brains, but not computers (e.g. Turing machines) could use mathematical intuition.

The distinction is summarised very briefly in [Turing\(1938\)](#), based on his 1936 Thesis. I suspect he had had insights related to Kant's three distinctions described below. I also suspect that Turing had a hunch that some ancient mathematical abilities of human brains make use of sub-neural chemical processes involving a mixture of continuous (folding, twisting, coming together or apart, etc.) and discrete processes (forming or releasing chemical bonds, with the properties described by Schrödinger). This combination can be thought of as a type of multi-branching Turing machine.

It remains to be shown that such a machine could support the examples presented below, of spatial mathematical reasoning relevant to claims implicitly made by Kant and Turing. I shall not attempt to demonstrate in detail that they really were making these claims. The claims are of interest whether made by Kant or Turing or neither. They are also potentially relevant to an enormous variety of biological control processes, including processes of development of complex organisms (including consumption and use of physical/chemical materials developed in other organisms) and also processes of perception, motor control, decision making, learning, mathematical discovery and scientific theorising. Moreover, insofar as all these mechanisms could not have existed before life existed, and could not all have come into existence simultaneously, it follows that theories about the mechanisms of biological reproduction and evolution will need to be extended to explain how enormously complex and varied mechanisms that exist now on this planet could have been products of much simpler and apparently uniform processes available before the formation of galaxies, solar systems, and planets. Much psychological, neuroscientific, social-scientific and philosophical literature will then have to be replaced by deeper, more accurate theories. One of the consequences may be a new lease of life for Kant's philosophy of mathematics in an expanded form, related to brain mechanisms of which he had no knowledge!

This is not a theory about "possible worlds"

There are modern philosophical and logical discussions of modal properties that I think are irrelevant to the points Kant and Turing were making. In particular, one modern notion starts from the concept of *a set of possible worlds*, and divides propositions into different categories according to which set of possible worlds (if any) they are true in. In this modern sense a proposition would be "necessary" if it is true in all possible worlds, "impossible" if it is false in all possible worlds", and "contingent" if it is true in some possible worlds but not all. However, I don't think these "possible world" concepts are relevant to the sense in which a child may discover that it is impossible to separate rigid linked rings made of impermeable materials, or that spatial containment is

necessarily transitive, i.e. if some object, or region of space, X is wholly contained in a region of space Y and Y is wholly contained in a region of space Z, then X is necessarily contained in Z.

As I tried to point out in [Sloman\(1962\)](#), thinking about such geometrical facts requires only the ability to think about *possible and impossible fragments of this world* and their relationships, not the ability to think about multiple total universes as assumed in "possible world semantics" for modal concepts.

I see no evidence that young children, or other intelligent animals able to recognize spatial impossibilities are thinking about all possible universes. (Certainly I was not when I learnt about geometrical impossibilities and necessities while studying geometry at school in the 1950s.) We can however, correctly say that ancient discoveries about possible or impossible configurations, including necessary truths of geometry and topology, are concerned with *possible and impossible configurations of possible fragments of this universe*.

There may be alternative possible universes in which space contains topological structures that do not exist in this world. Alternatively we may be mistaken about this world because it includes possibilities ruled out by ancient mathematical theories, as suggested by Einstein's Special and General theories of relativity, or by Quantum theory. On the other hand there may also be portions of this world that are correctly described by Euclidean geometry, even if the whole universe does not conform. The main point is that (suitably educated) human brains can identify such a type of world, or sub-world, and then go on to discover necessary features of such a world, using ancient forms of geometrical reasoning. In that context, necessary features are not features of all possible worlds.

The earliest discoveries, by ancient mathematicians, of truths and falsehoods concerning geometry made essential use of spatial reasoning, i.e. using diagrams and spatial operations on diagrams to prove that certain combinations of spatial properties were impossible, or to prove that having certain spatial properties necessarily implied having another spatial property, as in the triangle sum theorem mentioned above.

Pythagoras' theorem, also known to ancient mathematicians, states that if a triangle has an angle whose size is exactly 90 degrees (a quarter of a rotation), then the square on its longest side has an area equal to the sum of the squares on the other two sides. Ancient mathematicians discovered many different ways of proving such necessary truths of geometry and topology. But they were all considering only possible planar triangles in this world, not all possible complete worlds. (Barbara Vetter has made related points.)

How brains represent and reason about possible fragments of this universe is a deep question beyond the scope of current neuroscience (as far as I can tell). But perhaps a richer future version of neuroscience based on ancient biological mechanisms involved in gene expression, mentioned below, will one day yield new insights, answering questions raised by Turing about the nature of human mathematical *intuition* and its contrast with kinds of mathematical *ingenuity* that he regarded as implementable in digital computers.

This may also turn out to be related to some of Schrödinger's thoughts in *What is life?(1944)*. Biological structures can be built from molecules of many sorts, with different evolutionary histories, produced at different stages of individual development, providing an increasingly complex variety of both physical structures or mechanisms, and processes involving them, with many degrees and

kinds of complexity,

Process of evolution of genomes and processes of individual development can combine to produce multi-layered forms of control (some processes are controlled by others, which in turn are controlled by others, etc.) using chemistry-based mechanisms that evolved at different times, some of them based on molecular structures produced in other species that have to be consumed to provide required chemicals and chemical functions -- as Schrödinger noticed. Symbiotic relationships can also be used, instead of consumption. Moreover, analogues of symbiosis can occur within organisms insofar as parts or mechanisms evolved at different times are brought together to provide new functions.

Turing investigated machines that are not possible in a finite universe

If space and time in our universe are both finite (as suggested by some physical theories) that does not make it impossible for humans to think about possible structures and processes that cannot exist in this universe. For example, Turing thought about what we now describe as *Turing machines* that have infinitely long tapes (or tapes with infinitely many distinct locations where symbols can be added) that run for infinitely long times, or infinitely many time steps -- if each time step takes half the time of its predecessor then infinitely many time steps can occur in a finite time if the *resolution* of time is not bounded. (A similar point can be made about spatial infinity.) There is a substantial literature on possible worlds, most of which I am ignoring, as I don't regard it as relevant to the views of Kant or Turing. See the entry for [Possible Worlds](#) in the *Stanford Encyclopedia of Philosophy*.

Steps toward a solution [Remove?]

SHOULD THIS SECTION BE REMOVED BECAUSE TOO OBVIOUS?

Because the cut is planar and the polyhedron is convex, and the cut removes only one vertex, the cut must intersect each of the pre-existing planar surfaces meeting at the sliced off vertex. Each of those planar surfaces will have a triangular portion removed, leaving a new straight edge where the cut occurs.

All the new edges, visible or not, will together form the boundary of a new planar surface bounded by new straight edges meeting at newly created vertices. It must be planar because it is created by a plane surface through part of the polyhedron, cutting the polyhedron into two parts. (Only two parts because the polyhedron is convex.)

That new planar surface is in the plane of the cut: it separates the volume previously occupied by the sliced off material and the volume occupied by the remainder of the polyhedron.

If the original solid is shaped like a doughnut (a torus), then a single cut producing two separate parts can produce either a single new surface on each part, or, if the cutting edge straddles the hole in the doughnut, two new surfaces, but that has been ruled out by specifying that the original shape is a convex polyhedron. As far as I know, nobody understand what enables a human brain to think about these structures and processes there is not actual polyhedron or torus being sliced.

Readers may find it interesting to think about how the reasoning might be affected by considering a different view of the polyhedron to be sliced, namely one looking down at the vertex to be removed.

From that view all the edges and surfaces meeting at the vertex before it is sliced off will be visible, as will all the new edges and vertices and the new surface produced after slicing.

But given only the view shown in the picture above, it would be impossible to draw a picture of that "fully visible" view because the number of edges and surfaces originally meeting at the sliced off vertex is unspecified. Any drawing of the view from the top based on the original view depicted above, must therefore be incomplete.

Whichever views are considered, the process of thinking *spatially* about the problem is totally different from stating the problem using a logic-based axiom system for geometry and deducing the answer to the question using only logical reasoning, without any spatial reasoning.

Since visibility is nowhere mentioned in Euclid's axioms, and they include no mention of a slicing operation, I believe it would not be possible to solve the problem by giving a logical proof starting from Hilbert's axiom system for Euclidean geometry. It may be possible in an expanded version of Hilbert's system, based on new concepts added to Euclid's system. But the resulting geometric reasoning system would not be able to model the original reasoning processes used by non-expert mathematicians who can answer the question.

For anyone who has not worked out the solution: after the cut, (a) one old vertex will have been removed (the one sliced off), (b) each edge meeting at the old vertex will be shortened, producing new vertices at their new ends, (c) new edges joining the new vertices will bound newly produced polygonal faces, each of which is a truncated portion of an old face and (d) finally a new face is created in the plane of the cut, bounded by the new edges, meeting at newly formed vertices.

This form of "visual" reasoning easily leads to the conclusion that one vertex has gone, N new edges and N new vertices have been created, if there were N edges meeting at the removed vertex, and those new edges will bound a flat polygon with N edges and N vertices that did not exist before the cut. So there will be one new face, 1 vertex removed, N new vertices, and N new edges. Most people I have asked, discover that within a few minutes. However it took me about two years to realise that there is another case not covered by the above analysis, left as an exercise for the reader.

What brain mechanisms are required?

Working out that answer does not require unusual mathematical genius. I find that many people who have never previously encountered the problem, but understand all the concepts (especially "convex") can work out what the changes must be, even if they have never previously studied Euclidean geometry. Times taken may be a few seconds to several minutes, though a few individuals require longer times or cannot answer the question.

What sorts of brain mechanisms make this form of reasoning possible? It is not done simply by applying syntactic operations to discrete components of sentences, as in a logical theorem prover. It is also not done by examining a large number of polyhedra, cutting off one vertex with a planar cut, counting the changes, and then using statistical reasoning to work out the probabilities of various answers, as might be done by a neural-net based reasoning system. Unfortunately, neural nets are limited to deriving probabilities from statistical evidence: they cannot reason about what is *impossible* or what *necessarily* is the case. So they cannot make mathematical discoveries with the features specified by Kan.

As far as I know, no other species can make such a discovery. Why not? Moreover, very young humans cannot make such discoveries. Why not? What has to change in their brains between not being able to make them and being able to? I don't think anyone knows the answer at present. Unfortunately, most researchers haven't even thought about the question, including researchers in AI, psychology and neuroscience, because they have not encountered Kant's distinctions. I cannot offer an explanatory mechanism, except that it seems to require a reasoning medium that supports operations on spatial structures and the ability to abstract from the particular spatial structures and processes to produce generic answers to questions like ours.

I am not aware of any proposed mechanism in computer science, psychology, AI, neuroscience, or logic that can do the required reasoning. It could be done using logic if the statement of the problem included a rich description of the cutting process. But humans don't need that: they can work out the details by imagining a planar cut removing one vertex. I suspect the explanation depends on the use of sub-neural chemical mechanisms that can produce both discrete and continuous changes in representational structures -- a feature of chemistry that is missing from digital computers, making use only of rules allowing discrete changes with discrete consequences. But that feature of chemistry, made possible only by aspects of quantum physics (since molecular structures and processes could not exist in a Newtonian universe consisting only of interacting elastic particles acting under gravitational forces), was shown by Schrödinger in 1944 to be essential for biological reproduction mechanisms.

A meta-question

However, there is a related question: when you have found the answer, i.e. you can describe the changes produced by the slice, are you merely reporting *a generalisation from a collection of examples you have found*? Or have you made a deeper discovery, namely that that is what the answer *must* be, i.e. no other answer is *possible*? In other words have you discovered a *necessary* truth that is applicable to all *possible* ways of slicing a single vertex off any *possible* convex polyhedron, with a single planar slice?

Of course, very young children cannot understand the question, and some who do may not be able to answer it until they are older. That change does not require any training in answering questions like this, or experience of sawing through convex polyhedra. It may require some experience of seeing and manipulating solid objects, and experience of cutting off portions, in order to enable the concepts required for understanding the question to be developed. But the ability to answer the question requires a deeper kind of change: development of the ability to make non-empirical discoveries about relationships between spatial properties and processes.

How do you know that there isn't a convex polyhedron that you have never encountered, such that there is a way of removing exactly one of its vertices with a single planar cut that produces a different result?

I claim, inspired by Immanuel Kant's discussion in his *Critique of Pure Reason* that my example is one of infinitely many examples of possible discoveries of truths that are *synthetic* (non-analytic, non-definitional, not based on pure logic), that are *non-empirical*, i.e. not mere generalisations from examples and subject to refutation by an example that will be discovered one day, and *non-contingent*, i.e. *necessarily true*. (These three concepts, identified by Kant, are summarised and distinguished in [Sloman\(1965\)](#).)

There are infinitely many different convex polyhedra, and for each one infinitely many different ways in which exactly one vertex can be sawn off. But there is a fairly simple answer to the question how that process necessarily changes the numbers of vertices, edges and faces, and I claim that millions of humans are capable of understanding the question and discovering the answer, including realising that it is not merely an empirical generalisation that could have exceptions at high altitudes, or in the depths of an ocean.

What sort of brain development can enable a young child to acquire the ability to grasp that something is **impossible**, or is **necessarily** the case -- not just accidental results of slicing a particular vertex off a particular polyhedron?

There are many examples of such discoveries. E.g. by the age of five or six years many children seem to understand that one-to-one-correspondence is *necessarily* transitive, i.e. if there is a one-to-one correspondence between the members of two sets of objects, S1 and S2, and also a one-to-one correspondence between S2 and a third set, S3, then there *necessarily* exists a one-to-one correspondence between S1 and S3.

That fact is one of the discoveries that enabled our distant ancestors to discover the great utility of counting systems using a memorised collection of symbols to be used in different one-to-one correspondences, as explained in [Sloman\(1978\)](#) Chapter 8.

One consequence is that two collections of objects do not need to be adjacent and aligned for the existence of a 1-1 correspondence between them to be established. If both collections are in 1-1 correspondence with an initial sequence of a set of memorised numerals, then they must be in 1-1 correspondence with each other. This makes it unnecessary to take your whole family on a fishing trip to ensure that you catch at least one fish for each member, as pointed out in [Sloman\(2016\)](#).

Answering the question *why* a certain answer is correct involves describing a form of spatial reasoning that is sufficiently precise to produce the correct answer yet is independent of the numbers of vertices, faces and edges (V, F, E) involved or the precise locations or orientations of slicing operations that remove one vertex.

In other words, people who work out the answer are able to reason spatially not merely about a *particular* polyhedron with particular numbers V, E and F, e.g. a tetrahedron or a cube, but in a *general* way that applies to all possible initial convex polyhedra, and to any vertex sliced off, using any planar cut that removes only one vertex. Thus answering the question (as in many cases of geometric reasoning) involves a mode of thinking that correctly applies to infinitely many different structures and processes. This cannot be achieved by any form of probabilistic inference from statistical data, and therefore cannot be achieved by neural net based AI mechanisms.

I am deliberately leaving working out the answer as an exercise for the reader. After finding the answer, try to describe the reasoning you use and explain why it works no matter how many edges meet at the vertex chosen for removal.

This example, and many others, illustrate Kant's claim that it is possible to acquire mathematical knowledge about *necessary* (non-contingent) truths and falsehoods concerning possible types of structures and processes, including the spatial structures and processes investigated in Euclid's *Elements*. The proofs show "how things must be, or cannot be". When valid, they show why counter-examples to theorems are *impossible*.

For example, readers should be able to explain why their answer to the sliced polygon questions is true, by talking about the effects of the slice in a general way -- independent of the precise shape of the polygon. However, explaining how they do that, and what brain mechanisms make it possible, requires major advances in cognitive science/neuroscience/AI, based on (future!) deep theories about spatial cognition, its evolution, and its development in individual animals [Sloman\(2020\)](#).

There may be some people (e.g. students of David Hilbert?) who answer this and similar questions only by starting from a fully specified logical version of the problem, and then reason using only logic. But for most of the history of geometry that could not be done, until Hilbert (or a similar thinker) had produced a purely logical specification of Euclidean geometry. Moreover the mathematicians and non-mathematicians to whom I have presented the sliced polygon problem and other problems have all dealt with the problem by reasoning about spatial structures and processes, not logical structures and logical manipulations of formulae.

The third feature of mathematical knowledge/truth, i.e. necessity, is frequently omitted from summaries of Kant's claims about mathematical knowledge. Unfortunately, his ideas are now either ignored by most philosophers of mathematics or badly misrepresented, e.g. in [Hempel\(1945\)](#).

During the 20th Century, mathematical and philosophical opinions on the nature of mathematical knowledge changed, partly under the influence of developments in formal logic (by Boole, Peirce, Dedekind, Peano, Frege, Russell and many others). Many mathematicians regarded Hilbert's logic-based axiomatisation of Euclidean Geometry, [Hilbert\(1899\)](#) as removing the need for any non-logical forms of representation or reasoning in geometry.

xxxxxx REMOVEDxxxx

Moreover, Euclidean geometry had already been dethroned by a combination of discovery of alternative geometries, Einstein's theory of general relativity, claiming that physical space was not Euclidean, and Eddington's confirmation of Einstein's work based on observations of the solar eclipse in 1919.

After becoming friendly with philosophy graduate students in Oxford, around 1959, I learnt that philosophers thought that Immanuel Kant's philosophy of mathematics e.g. as presented in [Kant\(1781\)](#) had been refuted. A typical view of Kant as mistaken was expressed Hempel, referenced above. Yet Kant's view, as I understood it, corresponded to my experience as a student learning about geometry, making discoveries and finding proofs. So, around 1959, I switched from mathematics to philosophy in order to defend Kant's view of mathematical discovery, completing my thesis in 1962 [Sloman\(1962\)](#), partly summarised in [Sloman\(1965\)](#). I tried to show that, as Kant had claimed, the labels "Analytic", "Non-empirical" and "Necessarily true (or false)" indicate importantly different distinctions that can be applied to different kinds of knowledge.

Can quantum physics explain all these processes?

(INCOMPLETE NOTES)

When increasingly complex objects are formed by combining previously complex objects which take part in increasingly complex processes, the forms of representation that suffice for describing and explaining the fundamental/minimal structures and processes may be incapable of representing derived structures.

For example I suggest that when minimal objects are combined to form semi-rigid structures whose parts can change their relationships, and those parts are composed of semi-rigid structures whose parts can change their relationships, will the behaviours of the completed structure be describable and explainable in terms of the fundamental physical particles and their interactions?

Designing and controlling a dockside crane

Consider parts of a dockside crane, for example, that human designers and crane-drivers, etc. can think about and control in terms of changing relations between various "macro" parts and their relationships, e.g. a wheeled carriage at the bottom, able to move back and forth along dockside rails, with a tower mounted on the carriage, with a platform at the top, on which is mounted a cab on a platform that can rotate in a horizontal plane, with a jib whose angle with the horizontal can be increased or decreased, which guides a flexible cable by rotating the jib, so that when the cable is used to raise or lower the hook or grab suspended from the cable, the effects will depend on the slope of the jib, the plane of the jib, and the positions and relationships of other parts.

Is it possible for fundamental physical theory to represent all those constraints in such a way as to predict or explain all the movements of all the sub-atomic particles making up the whole structure? Whether that is possible or not, I think it is obvious that attempting to control, predict, and explain the behaviours of the crane as the various parts respond to forces or control signals would be completely intractable. Instead a controller should use the much smaller collection of facts about how the various "macro" parts of the machine are related and how those relationships change.

I suggest that a similar point applies to biological control systems controlling the behaviours of parts of organisms, e.g. bones, and muscles of limbs on a vertebrate. The same argument applies to much smaller control mechanisms involved in various stages of growing and controlling parts of an organism that starts as a single cell but gradually acquires more and more distinct parts that behave as new units in relation to other components. Attempting to control most parts of a developing organism by representing and modifying coordinates of the fundamental particles of which they are composed would be a completely intractable task. But by "factoring" that complexity into a relatively small collection of relationships between "macro" parts the control mechanisms need use and manipulate only information about those parts -- not the parts of the parts etc down to all the states and processes involving fundamental particles.

This suggests that in order to produce tractable control mechanisms in living organisms, biological evolution must have constantly produced new metaphysical layers, that allowed controllers operating on those layers to reach decisions in a reasonable time, and with reasonable representational resources, which would not be possible if ever control mechanism had to refer explicitly to every fundamental particle involved in the processes being controlled.

In other words in order that increasingly complex evolved organisms or parts of organisms can behave in biologically productive ways, evolution had to be metaphysically creative and produce mechanisms that acquired and used explicit information about newly created metaphysical layers. (I think this can be shown to illustrate Alastair Wilson's idea that "Grounding is metaphysical causation" [2017](#)).

This idea that the control mechanisms in biological organisms, as they control increasingly complex sub-systems during evolution, or during individual development, need to make use of information referring to new metaphysical layers of complexity is relevant to discussions in philosophy

regarding the reality of metaphysical distinctions made by humans, e.g. the distinction between physical states and processes in brains and mental states and processes. Far from the more abstract layers merely being human inventions for their own convenience they are products of biological evolution that play an essential part in the control of biological processes, such as reproduction, growth, digestion, tissue repair, waste disposal, infection control, and many kinds of ongoing maintenance. These ideas illustrated well in this video presentation: [Hoffmann\(2012\)](#).

I don't know how much of this Turing had thought about. By the time he died the biological understanding of the chemical bases of reproduction, development, physiological control, antibody production, etc. was only just beginning to get off the ground, following the discovery of the structure of DNA.

Perhaps he had already begun to think about such developments, and the 1952 paper was merely a publishable by-product of an unfinished, much deeper, long term research project, that would later have included explanations of the importance of chemistry in brains, underpinning multiple evolved metaphysical layers of information and control.

The Conway knot problem

A fairly recent (and complex) example of spatial intelligence is the work of a graduate mathematics student, Lisa Piccirillo, who found a new way of thinking about a famous old unsolved problem in topology *Whether the Conway knot -- discovered years ago by the mathematician John Conway, inventor of the "Game of Life" -- is a "slice" of a higher-dimensional knot*. An online video gives more information:

<https://www.quantamagazine.org/graduate-student-solves-decades-old-conway-knot-problem-20200519/>

I have not looked closely enough, or thought about this long enough, to understand either the problem or the solution in any detail, but it seems to be a clear example where logic and empirical investigation did not suffice to provide a solution to the problem. It required a powerful visual imagination and familiarity with a subspace of spatial transformations.

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<https://www.birmingham.ac.uk/staff/profiles/computer-science/tino-peter.aspx>

Anthony Leggett

<https://physics.illinois.edu/people/directory/profile/aleggett>

Alastair Wilson

<https://www.birmingham.ac.uk/staff/profiles/philosophy/wilson-alastair.aspx>

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<https://www.canterbury.ac.nz/arts/contact-us/people/jack-copeland.html>

APPENDIX: Varieties Of Virtual Machinery

I suspect that if Schrödinger had lived a few decades longer he would have recognized that the concept of a *virtual machine*, provides the required type of answer to his mystification about consciousness in living organisms.

The label "virtual machine" was originally used to refer to a computer program running on a physical machine, to test the design for a proposed but not yet built physical machine. In that sense a virtual machine was a single program running on a single physical machine emulating a possible but non-existent physical machine -- e.g. to be built with a new hardware design.

Over time, that notion was extended in various ways and became increasingly general. Nowadays we are familiar with virtual machines, which, far from being tied to a particular physical machine while running can be distributed over many physical machines linked in a network, and can continue running while the physical machines are changed either by addition of new machines replacing old ones or simply by adding new physical machines, e.g. to increase the speed and memory capacity of the virtual machine, or to add new components (e.g. new security mechanisms or new types of information stores) implemented in new hardware, new software, or a combination. Examples of such "distributed" virtual machines that endure across a history of changing implementations are banking systems, email systems, information systems like Google, and marketing and selling systems like Amazon, and many more. Despite the label "virtual" these machines are in no sense unreal, or provisional test systems. They also have real causal powers, which can be of great benefit to owners, or users, or in some cases great sources of nuisance. Internet

A skeptic (or Schrödinger) might object that such virtual machines cannot experience pains or desires. But the answer to that would be to explain how a complex control system might have a variety of devices for detecting sources of malfunction or potential error and start signalling warnings to a more central control system, which will be able to assess the severity of the warning and decide whether to take immediate action or postpone dealing with the problem. However the detectors may have the ability to judge that the situation is getting worse with a need for urgent remedial action, and repeatedly send interrupt signals to the higher level control system, which may reach a stage where those signals are too disruptive to be ignored any longer, so that it then directs resources to attend to the source of the problem. I have summarised what could be the thought processes of a designer of such a system. However evolutionary processes could, and biological evolution does, produce mechanisms that have no designer, yet function as if they had. For more on this line of argument, and the concept of "Virtual machine functionalism" see: [Sloman\(2013\)](#) and [Sloman & Chrisley\(2003\)](#) and other references in those documents.

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