School of Computer Science — University of Birmingham UK

The Triangle Sum Theorem

Old and new proofs concerning the sum of interior angles of a triangle. (More on the hidden depths of triangle qualia.)

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http://www.cs.bham.ac.uk/~axs

NOTE ON MATHEMATICAL EDUCATION

Updated 11 Oct 2022

I have discovered that a large proportion of highly intelligent, mathematically sophisticated, researchers have had no personal experience of finding geometrical constructions and proofs, which used to be a standard feature of mathematical education in schools until around the middle of the 20th century, when such teaching was replaced by an emphasis on use of formal, rigorous, logic-based reasoning and set theory, as recommended by the Bourbaki movement. See
https://en.wikipedia.org/wiki/Nicolas_Bourbaki
https://www.encyclopedia.com/science/encyclopedias-almanacs-transcripts-and-maps/bourbaki-school-mathematics

As a result, there are now many highly intelligent, well educated researchers in many disciplines, including mathematics, psychology, neuroscience, philosophy and AI, who have never encountered mathematical applications of ancient forms of spatial reasoning that used to be a standard part of mathematical education, and which cannot be replicated using standard AI mechanisms based solely on digital computation, and therefore do not allow mechanisms in which shapes, sizes, angles, and relative distances are continuously transformed.

Moreover, those ancient forms of mathematical discovery cannot be based on "Neural network" (NN) mechanisms that function by collecting statistical evidence and then deriving probabilities, whose influence can be propagated across links in neural networks. Such NN mechanisms can never be used to prove that something is necessarily true or necessarily false (impossible) because necessity and impossibility cannot be derived from statistical data: only probability (probable truth or probable falsehood) can be derived.

Yet, ancient mathematicians used geometric reasoning to establish impossibility or necessity, for example that the internal angles of a planar triangle **necessarily** sum to 180 degrees (a straight line); and Pythagoras' theorem: if squares are constructed on the sides of a right-angled planar triangle then the area of the square on the hypotenuse (the longest side) is **necessarily** equal to the sum of the areas of the squares on the other two sides of the triangle.

I am not the only person who regards the failure to teach the ancient spatial, or diagrammatic, modes of reasoning as an educationally disastrous mistake. For example, Benoit Mandelbrot, who made major contributions to the study of fractal geometry, shared that opinion. For information about fractals see:

https://www.ibm.com/ibm/history/ibm100/us/en/icons/fractal/.

Installed: 9 Sep 2012
Please report bugs to: A.Sloman@cs.bham.ac.uk
This was originally part of the file:
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/triangle-theorem.html
That file is mainly about areas, so this portion, concerned with angles,
was moved here on 28th May 2013.

Last updated:
10 Oct 2022 (modified note on mathematical education above)
2 May 2020 then 22 Aug 2022 (minor re-organisation and re-formatting)
26 May 2019 (minor re-formatting).
5 Apr 2018: Earlier version of Pardoe proof by Thibaut (1809) referenced below.
25 Apr 2016; 8 Sep 2017; 23 Sep 2017;
26 Feb 2015: added link to document showing how in P-geometry
an arbitrary angle can be trisected (impossible in pure Euclidean geometry):
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/trisect.html
29 May 2013 ....Updates: part of the original file deleted...

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This file is
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/triangle-sum.html
PDF derivative:
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/triangle-sum.pdf

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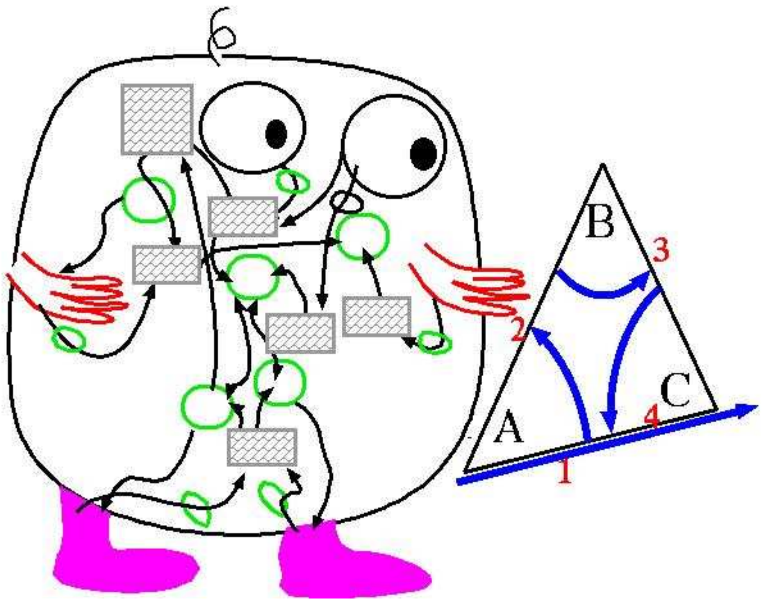

When will the first baby robot grow up to be a mathematician?

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The Triangle Sum Theorem

The triangle sum theorem is normally expressed as "The interior angles of a triangle add up to 180 degrees". This assumes a standard way of measuring angles, according to which a complete rotation would be 360 degrees and a half rotation 180 degrees. But we can equivalently express the theorem as "The interior angles of a triangle add up to a straight line", or "... half a rotation", which does not require any conventional unit for measuring angles".

As we'll see below, that suggests a way of proving the theorem by considering a succession of rotations and noticing what they add up to, an idea suggested to me many years ago by a former student, Mary Pardoe, after she had become a mathematics teacher.

One of the standard ways of proving the following theorem is presented below:

Triangle Sum Theorem (TST):
The interior angles of a triangle add up to a straight line, or half a rotation (180 degrees).

The standard methods of proof all make use of some version of Euclid's parallel postulate, (Axiom 5 in Euclid's elements) which can be formulated in several equivalent ways, e.g.

Definition:
Two straight lines L1 and L2 are parallel if and only if they are co-planar and have no point in common, no matter how far they are extended.

Postulate:
Given a straight line L in a plane, and a point P in the plane not on L, there is exactly one line through P that is in the plane and parallel to L. (That was not Euclid's formulation, but is perhaps intuitively the clearest formulation.)

All of this presupposes the concept of "straightness" of a line. For now I'll take that concept for granted, without attempting to define it, though we can note that if a line is straight it is also symmetric about itself (it coincides with its reflection) and also it can be slid along itself without any gaps appearing. If it were possible to view a straight line from one end it would appear as a point, in an environment in which light is not "bent" by a strong gravitational field, or a change in the material through which it passes.
[I am glossing over a number of problems about definability of geometrical properties of lines, surfaces, etc.]

The "standard" ways of proving the Triangle Sum Theorem make use of properties of angles formed
when a straight line joins or crosses a pair of parallel lines. E,g,

COR: Corresponding angles are equal:
If two lines L1, L2 are parallel and a third line L3 is drawn from any point P1 on L1 to a point P2 on L2 and continued beyond P2,
then the angle that L1 makes with the line L3 at point P1, and the angle L2 makes with the line L3 at point P2 (where the angles are on the same side of both lines) are equal.

ALT: Alternate angles are equal:
If two lines L1, L2 are parallel and a third line L3 is drawn from any point P1 on L1 to a point P2 on L2,
then the angle L1 makes with the line L3 at point P1, and the angle L2 makes with the line L3 at point P2 (on the opposite sides of both lines) are equal.

For more on transversals and relations between the angles they create, see
http://www.mathsisfun.com/geometry/parallel-lines.html
That page teaches concepts with some interactive illustrations, but presents no proofs.

Standard Euclidean proofs of COR and ALT are presented here:
https://proofwiki.org/wiki/Parallelism_implies_Equal_Corresponding_Angles
https://proofwiki.org/wiki/Parallelism_implies_Equal_Alternate_Interior_Angles

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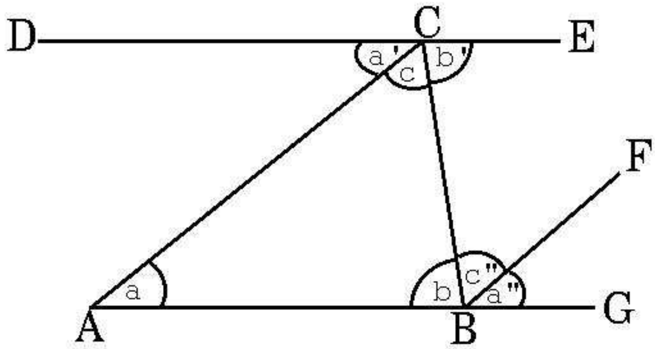
The "standard" proofs of the "Triangle Sum Theorem"

Two "standard" proofs of the triangle sum theorem using parallel lines, and the Euclidean theorems **COR** and/or **ALT** stated above, are shown below in Figure Ang1:

To be proved:
In triangle ABC the interior angles, a, b, c sum to a straight line.



Proof 1: (Does not use line BF or angles a" c")
Line DE is parallel to line AB, and AC is a transversal joining them. Since a and a' are alternate angles, they must be equal; likewise BC is a transversal joining parallel lines AB and DE, with b and b' alternate angles and therefore equal.

So: a+b+c = a'+b'+c Q.E.D

Proof 2: (Does not use line DE or angles a' b')
Line BF is drawn parallel to AC. Line AB, extended to G is a transversal crossing parallel lines AC and BF. So a and a" are corresponding angles and therefore equal. CB is another transversal joining the two parallel lines AC and BF with c and c" alternate angles and therefore equal.

Figure Ang1: So: a+b+c = a"+b+c" Q.E.D

Warning: I have found some online proofs of theorems in Euclidean geometry with bugs apparently due to carelessness, so it is important to check every such proof found online. The fact that individual thinkers can check such a proof is in part of what needs to be explained.

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Mary Pardoe’s proof of the Triangle Sum Theorem

Many years ago at Sussex university I was visited by a former student Mary Pardoe, who had been teaching mathematics in schools. She told me that her pupils had found the standard proof of the triangle sum theorem hard to take in and remember, but that she had found an alternative proof, which was more memorable, and easier for her pupils to understand.

> **Note:** In the original publication reporting this proof I mistakenly referred to the author as Mary Ensor, her name as a student. I think she was already Mary Pardoe at the time she visited me.

Her proof just involves rotating a single directed line segment (or arrow, or pencil, or ...) through each of the angles in turn at the corners of the triangle, which must result in its ending up in its initial location pointing in the opposite direction, without ever crossing over its original orientation.

So the total rotation angle is equivalent to a straight line, or half rotation, i.e. 180 degrees, using the convention that a full rotation is 360 degrees.

The proof is illustrated below in Figure Ang2.

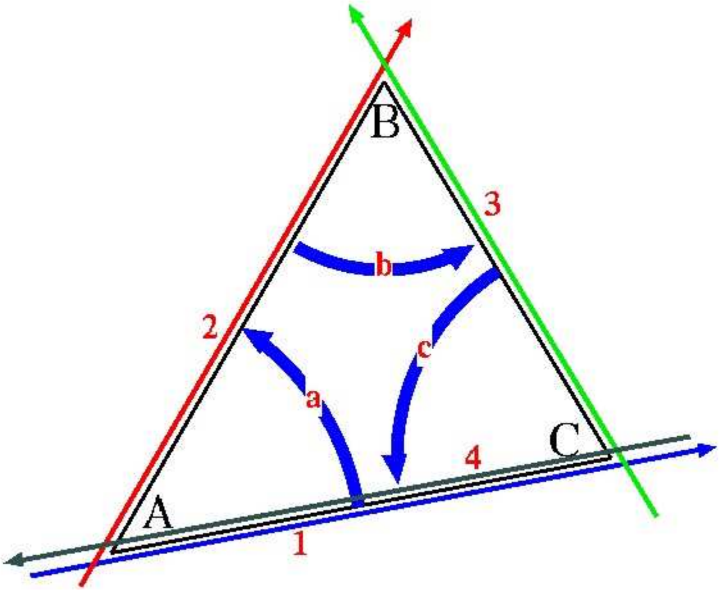


Figure Ang2:

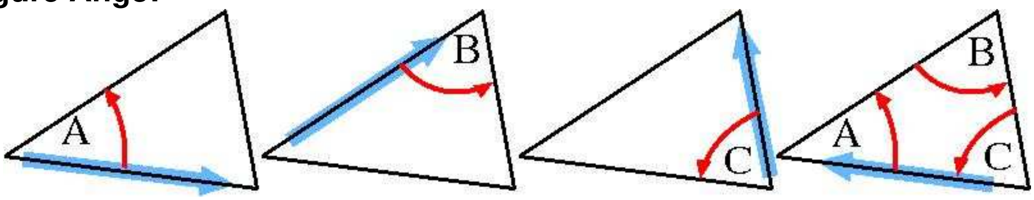
In order to understand the proof, think of the blue arrow, labelled "1", as starting on line AC, pointing from A to C, and then being rotated counter-clockwise, first around point A, then around point B, then around point C until it ends up on the original line but pointing in the direction of the dark grey arrow, labelled "4".

So, understanding the proof involves considering what happens if

- the blue arrow labelled "1" initially lies on the side AC of the triangle,
- then is rotated counter-clockwise through angle **A**, indicated by the curved arrow labelled "a", to the location of the red arrow labelled "2", assumed to lie along the side AB of the triangle,
- then rotated counter-clockwise through angle **B**, as indicated by curved arrow "b", to the location of the green arrow labelled "3", assumed to lie along the side BC of the triangle,
- then rotated counter-clockwise through angle **C**, as indicated by curved arrow "c", to the location of the dark grey arrow labelled "4" assumed to lie along the side CA of the triangle.

A "time-lapse" presentation of the proof may be clearer, as shown in Figure Ang3 (which may not work on all operating systems).

Figure Ang3:



It is best to think of the proof not as a static diagram but as a **process**, with stages represented from left to right in Figure Ang3. In the first stage, the pale blue arrow starts on the bottom side of the triangle, pointing to the right then is rotated through each of the internal angles A, B, C, always rotated in the same direction (counter-clockwise in this case), so that it lies on each of the other sides in succession, until it is finally rotated through the third angle, c, after which it lies on the original side of the triangle, but obviously pointing in the opposite direction. Some people may prefer to rotate something like a pencil rather than imagining a rotation depicted by snapshots.

In this triangle the sides are not very different in length, which conceals a problem that can arise if the first side the arrow is on is very short and the other two are much longer. If the length of the rotating arrow is fixed by the length of the first side, you would need to imagine either that the arrow stretches or shrinks as it rotates, or that it slides along a line after reaching it so as to be able to rotate around the next vertex. Alternatively you can imagine that the depicted arrow is part of a much longer invisible arrow, so that, as the invisible arrow rotates from one side to another, it always extends beyond both ends of the new side, and can then rotate around the next vertex. I leave it to the reader to think about these alternatives and what difference they make to the proof, and to the cognitive competences required to construct and understand the proof.

For an arrow to be rotated in a plane and end up lying in its original position it must have been rotated through some number of half-rotations. (Each half rotation brings it back to the original orientation, but pointing in alternate directions.)

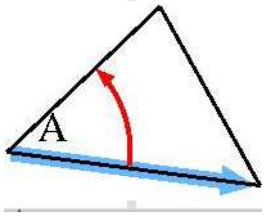
Since (1) the arrow at no point crossed over its original orientation, and (2) it ended up pointing in the opposite direction to its original orientation, the total rotation was through a half circle -- which is clear if you actually perform the rotations using a physical object, such as a pencil.

And since that rotation was made up of combined rotations through angles A, B, and C, those three angles must add up to a half circle, i.e. 180 degrees.

A crucial feature of our ability to think about the diagram and the process, is that we (presumably including you, the reader) can see that the key features of the process could have been replicated, no matter what the size or orientation of the triangle, no matter what the lengths of the sides or the sizes of the angles, no matter which side the arrow starts on, no matter which way it is pointing initially, and no matter in which order the rotations are performed, e.g. A then B then C, or C reversed, then B reversed, then A reversed.

This proof of the triangle sum theorem, using a rotating moving arrow, works for all possible triangles on a plane -- as do the standard Euclidean proofs using parallel lines.

MOVIE PROOF OF TRIANGLE SUM THEOREM
(May not work on all web browsers.)



On a plane surface, rotating the blue arrow through the three internal angles (i.e. A, then B, then C) always brings it back to the starting line, pointing in the reverse direction, without ever crossing over its original orientation, and this (obviously?) doesn't depend on the shape of the triangle.

This proof is unlike standard proofs in Euclidean geometry since it involves consideration of continuous processes, and therefore involves time and temporal ordering, whereas Euclidean geometry does not explicitly mention time or properties of processes -- though there are some theorems about the locus of point or line satisfying certain constraints, which can be interpreted either as specifying properties of processes extended in time, or as properties of static trajectories, e.g. properties of lines or curves.

NOTE:

http://www.cs.bham.ac.uk/research/projects/cogaff/misc/p-geometry.html
presents a more detailed, but still incomplete, discussion, of the geometrical prerequisites for some of the above reasoning. It introduces the idea of P-geometry, which is intended to be Euclidean geometry without the Axiom of Parallels (Euclid's Axiom 5), but with time and motion added, including translation and rotation of rigid line-segments.

NOTE (Added 8 Sep 2017):
Mary Pardoe remains actively involved in mathematics education.
Her twitter site is a steady stream of information: https://twitter.com/pardoemary
Some of Vi Hart's wonderful mathematical video doodles are also relevant: http://vihart.com/

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Is the Pardoe proof valid?

NOTE: I have presented Mary Pardoe's proof in several places, over several years, e.g.

Aaron Sloman, 2008,
Kantian Philosophy of Mathematics and Young Robots, in *Intelligent Computer Mathematics*, Eds. Autexier, S., Campbell, J., Rubio, J., Sorge, V., Suzuki, M., and Wiedijk, F., LLNCS no 5144, pp. 558-573, Springer,
http://www.cs.bham.ac.uk/research/projects/cosy/papers#tr0802
(This paper referred to Mary Ensor.)

Aaron Sloman, 2010,
If learning maths requires a teacher, where did the first teachers come from?, In *Proceedings Symposium on Mathematical Practice and Cognition,* AISB 2010 Convention, De Montfort University, Leicester
http://www.cs.bham.ac.uk/research/projects/cogaff/10.html#1001

And in talks on mathematical cognition and philosophy of mathematics here:
http://www.cs.bham.ac.uk/research/projects/cogaff/talks/

The presentations produced no responses -- either critical or approving, except that in one informal discussion a mathematician objected that the proof was unacceptable because the surface of a sphere would provide a counter example. However, the surface of a sphere provides no more and no less of a problem for Pardoe's proof than for the standard Euclidean proofs since both proofs are restricted to planar surfaces.

I tried searching for online proofs to see if anyone else had discovered this proof or used it, but nothing turned up. The proof using rotation is so simple and so effective that both Mary Pardoe and I feel sure it must have been discovered previously.
(**Added 19 Oct 2018:** It was discovered previously! See the note about Thibaut below.)

NOTE ADDED 6 Oct 2012 (Asperti and Scott):

I have discovered that as a result of the discussion in 2010 on the MKM-IG email list, Andrea Asperti mentioned the proof (and the email discussion) in this paper, discussing related issues:

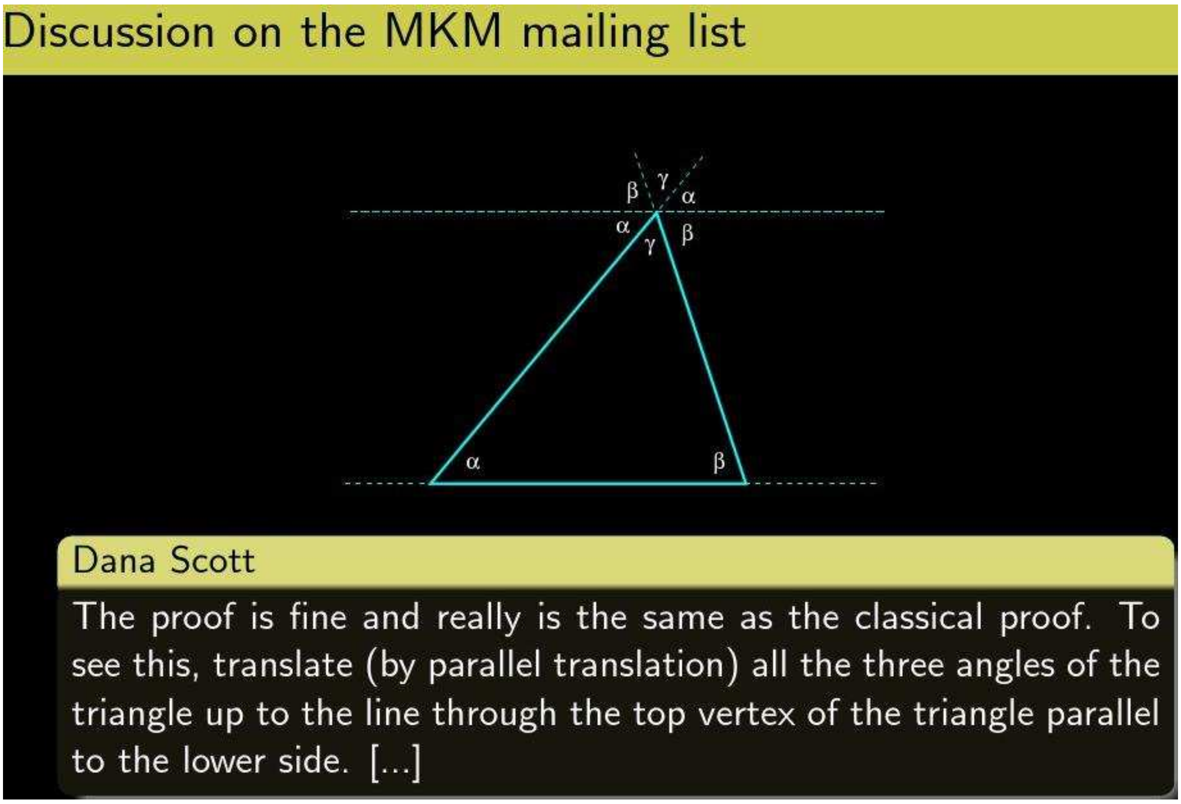
Andrea Asperti, Proof, Message and Certificate,
in *AISC/MKM/Calculemus*, 2012, pp. 17--31,
Online: http://www.cs.unibo.it/~asperti/PAPERS/proofs.pdf
http://dx.doi.org/10.1007/978-3-642-31374-5_2

And in this slide presentation with the same title, starting with Mary's proof, and comments on the proof by Dana Scott and Arnon Avron:

http://www.cs.unibo.it/~asperti/SLIDES/message.pdf

Dana Scott commented:

"The proof is fine and really is the same as the classical proof. To see this, translate (by parallel translation) all the three angles of the triangle up to the line through the top vertex of the triangle parallel to the lower side."



and later added

> "I should have commented in my explanation of the proof that if you translate the line on which the base of the triangle sits along each of the sides up to the vertex, then both actions result in the same line - the unique parallel."

Arnon Avron wrote:

> If this "proof" is taught to students as a full, valid proof, then I do not see how the teacher will be able to explain to those students where the hell Euclid's fifth postulate (or the parallels axiom) is used here, or even what is the connection between the theorem and parallel lines.

Is it really the same as the classical proof?

But this still leaves several questions open: Why did neither Mary, nor her students, nor others to whom I have shown the proof not claim that they could accept it ONLY by relating it to a proof that depends explicitly on Euclid's parallel axiom; and why should anyone regard as inferior the direct intuition provided by this proof that the three internal angles add up to half a rotation?

I suspect we shall not have good answers to these questions until we have a much deeper understanding of the combination of biological geometric reasoning mechanisms produced by evolution plus the (epigenetic) processes of individual development leading up to use of those mechanisms -- deep enough to build a baby robot that can grow up to have the competences of ancient mathematicians.

We must not forget that however those competences are eventually explained they were of tremendous importance for human beings, not least because the contents of Euclid's *Elements* are still in use by engineers, scientists and mathematicians all around the planet, every day.

What is the cognitive function of a mathematical proof?
Added 16 Aug 2018

As for what exactly the "function" of a proof is, I suspect that this will not be clear until we understand better the evolutionary origins of the mechanisms for discovering, understanding, and using ancient proofs -- whose role was totally different from modern conceptions of proof as depending on and conforming to logical reasoning mechanisms that were mostly developed long after Archimedes, Euclid, Zeno and others made their discoveries.

In particular, it is clear that all the axioms and postulates of Euclid's *Elements* were originally *discoveries* not arbitrarily selected starting points for chains of reasoning, even if Euclid can be interpreted as presenting them as if they were.

It is also important, as Kant observed, that discoveries in mathematics are characterised by being about necessity and impossibility (two sides of the same coin since what's impossible is what's necessarily false). The biological importance of this is that animals that can classify describable structures and processes and impossible, or as having necessary consequences, have an enormous biological advantage over those that have to check everything by collecting masses of statistical evidence and reasoning probabilistically. For samples of practical uses of such abilities see
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/changing-affordances.html

Predicting Affordance Changes
and
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/toddler-theorems.html
Toddler Theorems: Case Studies

The kinds of learning, discovery, and practical use of these topological and geometric facts are beyond the scope of current (e.g. 2017) AI robot designs and learning mechanisms, e.g. "deep learning" that depends on probabilistic reasoning, which can never establish necessity or impossibility. (This topic is discussed further in
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/ijcai-2017-cog.html)
See also this draft discussion of some of the roles of compositionality in biological evolution and its products:
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/sloman-compositionality.pdf
(An html version may be added later.)

More on P-Geometry

The P-geometry document mentioned above begins to specify a variant of Euclidean geometry without the parallel axiom, but allowing for translation and rotation of line segments while maintaining their length.
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/p-geometry.html

P-geometry (not yet fully specified) is used to trisect an arbitrary angle.
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/trisect.html

NOTE:
There is a "process" version of the proof of Pythagoras theorem that makes use of a video. A version implemented in Pop-11 is illustrated in the video in this tutorial:
http://www.cs.bham.ac.uk/research/projects/cogaff/tutorialspythagoras.html
The video attempts to demonstrate the invariance by showing how the shapes and or sizes of the triangles, squares and rectangles can be changed without changing the structural relationships. This was inspired by a demonstration originally provided by Norman Foo, using different transformations:
http://www.cse.unsw.edu.au/~norman/Pythag.html
One of the striking facts about Pythagoras' theorem is how many different ways it can be, and has been, proved.

NB: The programs that present such proofs do not themselves understand the proofs. They can be powerful "cognitive prosthetics" for humans learning mathematics, but the programs do not know what they have done, or why they have done it, and do not understand the invariants involved -- e.g. essentially the same proof could have started with a triangle with different angles, or a triangle of a different size.

An earlier discovery of Mary Pardoe's proof
Added 5 Apr 2018

On 26 Mar 2018, Tim Penttila (School of Mathematical Sciences, The University of Adelaide) wrote to me with very interesting information about the history of this proof. This is what he wrote:

The proof of the angle sum of a triangle that you attribute to Mary Pardoe was first published by Bernhard Friedrich Thibaut (1775-1832) in the second edition of his *Grundriss der reinen Mathematik*, published in Goettingen by Vandenhoek und Ruprecht in 1809 (see page 363).

It is not valid without assuming an equivalent of the parallel postulate. In Euclidean geometry, the composition of three rotations by (directed) angles adding up to an integer multiple of a full turn is a translation; but this fails to be true without the parallel postulate. Thibaut had put forward the proof as part of an attempted proof of the parallel postulate; his attempted proof is discussed in Roberto Bonola's Non-Euclidean geometry: a critical and historical study of its development (page 63), in William Barrett Frankland's *Theories of parallelism: an historical critique* (page 37), and in Jean-Claude Pont's *L'aventure des paralleles histoire de la geometrie non-Euclidenne: Precurseurs et attardes* (pages 240-244).

Thus the proof is only valid for plane geometry where the plane is assumed to have the properties that it does in Euclid's *Elements*; it does not hold for the hyperbolic plane of Bolyai and Lobachevsky (which satisfies all those properties bar the parallel postulate). (This is likely why the objection about the surface of a sphere was raised to you.)

The objection that the surface of a sphere provides a counterexample is also over a century old, going back to Olaus Henrici's criticism of Thibaut's proof in "The axioms of geometry", published in *Nature*, Volume 29, 1884, pp.453-454 and 573.

Reply to Tim Penttila:
I am very grateful for this information. A small point of clarification, regarding this comment:
"It is not valid without assuming an equivalent of the parallel postulate."

That is exactly why some years ago I began, but did not finish, an exploration of the possibility of revising Euclidean geometry, as mentioned above, by replacing the parallel postulate with an axiom related to rotating and translating line segments, which I called (provisionally) "P-geometry" to reflect the inspiration of Mary Pardoe. My incomplete discussion is here:
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/p-geometry.html

The possibility of alternative axiomatic presentations of Euclidean geometry is a reflection of the fact that we have some deeper pre-axiomatic understanding of space, that allows us to discover truths that can be organised in terms of axioms, proofs and theorems. All presentations of Euclidean geometry explicitly or implicitly start with some set of axioms/postulates. As in many other fields of mathematics those axioms are not arbitrarily selected collections of symbols, but reflect mathematical discoveries that provide important facts from which other facts can be inferred. However those axioms are not *uniquely* correct starting points. They are all discoveries based on something deeper that, as far as I know has never been accurately identified. It must have been a product of biological evolution. See the Meta-Morphogenesis project for more on this:
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/meta-morphogenesis.html

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Added 9 Feb 2013: Another proof of the sum theorem, by Kay Hughes

Modified 4 Mar 2013:

In February 2013, at an orienteering club dinner, I was talking about education with Kay Hughes and asked if she could remember how to prove the Triangle Sum Theorem. She could not remember a proof, but quickly thought up a proof that I had never previously encountered. My presentation here does not use her words, but offers a more explicit elaboration of the ideas she presented, after a few minutes sitting silently, thinking, at the dinner table.

Her key idea was to use a theorem (which she apparently re-discovered!) about the sum of external angles of a polygon always being a whole rotation (360 degrees) and combining that with the fact that each of the *external* angles has an *internal* angle as complement, as explained in more detail below.

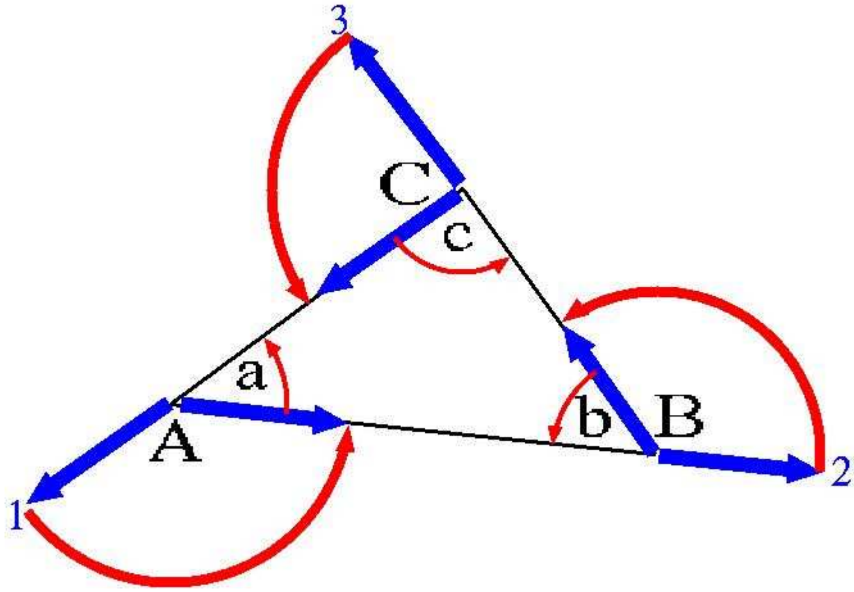

Figure Ang4:

It should be obvious from the figure that it presents a proof that the exterior **anti-clockwise** angles of a triangle (A+B+C) sum to a circle (360 degrees) as do the exterior **clockwise** angles, not shown in the figure.

Added 19 Mar 2013: This was named "The total turtle trip theorem" by Seymour Papert, in his **Mindstorms: Children, Computers, and Powerful Ideas** (1978), though it was well known long before then. (It can be generalised to smooth simple closed curves. See also http://en.wikipedia.org/wiki/Total_curvature .)

The exterior anti-clockwise angles are those obtained by extending each side in turn in one direction then rotating the extension to line up with the next side. So, for example, in Figure Ang4, the *internal* angles are **a**, **b** and **c**; whereas the *exterior* anti-clockwise angles **A**, **B** and **C** are got by extending the first side to location **1** then rotating the extension through angle **A** to the next side, then extending that side to location **2** and rotating the extension through angle **B** to the second side, and so on.

Because results of all those rotations bring the rotated arrows back to the original orientation, indicated at **1** in the figure, and the rotated arrow does not pass through its original direction, the total external anti-clockwise rotation must be a full circle (i.e. 360 degrees). An exercise left to the reader is to show that that's true not only for triangles but for all polygons, and, by symmetry, must also be true for the sum of the clockwise external angles.

> **Note**: There is a related "visual" proof posted here https://twitter.com/thingswork/status/1121857148068065280 (drawn to my attention by Ron Chrisley), based on what happens if a polygon shrinks to a point. This version applies only to polygons and does does not generalise (smoothly) into a proof that a tangent arrow moving around **any** simple closed curve, back to its starting point must have a resultant rotation of 360 degrees.

Returning to Fig. Ang4, above, for the special case of a triangle, Kay Hughes argued as follows
Theorem External: A + B + C = 360

But each of the internal angles is the **complement** of the adjacent internal angle, because they sum to a straight line. So we have these three truths:

A + a = 180 therefore **a = 180 - A**
B + b = 180 therefore **b = 180 - A**
C + c = 180 therefore **c = 180 - A**

So, the sum of the internal angles is
a + b + c = (180 - A) + (180 - B) + (180 - C)
= 180 + (180 + 180) - (A + B + C)
= 180 + 360 - (A + B + C)
Then substituting from **Theorem External:**
= 180 + 360 - 360
= 180
So, we have another proof of the standard Triangle Sum Theorem:
Theorem Internal: a + b + c = 180

Compare this with the video proof for the external angles posted on Twitter mentioned above: https://twitter.com/thingswork/status/1121857148068065280. This does not generalise easily to the corresponding "total turn theorem" concerning an arbitrary closed, non-self-crossing route in a plane.

I tried searching online for a version of Kay's proof of the triangle sum theorem using Fig. Ang4, and did not find a previous occurrence, though many web sites mention both the triangle sum theorem for interior angles and the theorem about exterior angles always summing to 360.

NOTE (Added 25 Apr 2016): Michael Fourman
(https://en.wikipedia.org/wiki/Michael_Fourman)
informs me that he encountered the external angle proof while at school.

Nerlich on geometry and metaphysics
Added 22 Jul 2018

I have just stumbled across this paper:

Graham Nerlich, 1991, How Euclidean Geometry Has Misled Metaphysics, *The Journal of Philosophy,*
88, 4, Apr, 1991 pp. 169--189,
http://www.jstor.org/stable/2026946

It points out that a common philosophical argument, namely that if everything were to gradually double in linear dimensions during a period of time that would be undetectable, and therefore there is no such thing as absolute size, breaks down if space is non-Euclidean.

This has implications for the status of the Pardoe proof, but also the status of other proofs in Euclidean geometry. I may later add some comments about that here. See also the discussion of P-geometry:
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/p-geometry.html

Related documents

A partial index of discussion notes in this directory is in
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/AREADME.html

See also this discussion of "Toddler Theorems":
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/toddler-theorems.html
(Or http://goo.gl/QgZU1)
These examples of varieties of necessity and impossibility are closely related:
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/impossible.html

There is a draft, incomplete, discussion of transitions in information-processing in biological evolution, development, learning, etc. here. That document and this one are both parts of the Meta-Morphogenesis project, partly inspired by Turing's 1952 paper on morphogenesis.

Gibson's theory of perception of affordances

James Gibson's theory of perception of affordances, is very closely related to mathematical perception of structures, possibilities for change, and constraints on changes (structural invariants). Gibson's ideas are summarised, criticised and extended here:
http://www.cs.bham.ac.uk/research/projects/cogaff/talks/#gibson

This discussion of theorems about processes that alter or preserve areas of triangles is closely related: http://www.cs.bham.ac.uk/research/projects/cogaff/misc/triangle-theorem.html

These discussions draw attention to common confusions about the nature of embodied cognition in 'enactivist' theories, and illustrate the need to distinguish 'online intelligence' from 'offline intelligence'.

Related Video On Adam Ford's Web Site
At the AGI conference in Oxford, December 2012, Adam Ford interviewed me about this and related topics. I used the triangle sum theorem as an example in the interview, available at
http://www.youtube.com/watch?v=iuH8dC7Snno

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Acknowledgements

See the acknowledgements section of the paper on P-Geometry
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/p-geometry.html#acknowledge

Offers of help in making progress will be accepted gratefully, especially suggestions regarding mechanisms that could enable robots to have an intuitive understanding of space and time that would enable some of them to rediscover Euclidean geometry, including Mary Pardoe's proof.

I believe that could turn out to be a deep vindication of Immanuel Kant's philosophy of mathematics. Some initial thoughts are in my online talks, including

http://www.cs.bham.ac.uk/research/projects/cogaff/talks/#toddler
Why (and how) did biological evolution produce mathematicians?

http://www.cs.bham.ac.uk/research/projects/cogaff/misc/ijcai-2017-cog.html
Video presentation with online notes:
Why can't (current) machines reason like Euclid or even human toddlers?
(And many other intelligent animals)
Prepared for AGA Workshop at IJCAI 2017.

Video of presentation at Oxford Mathematical Institute Conference on *Models of Consciousness*, Sept 2019:
https://www.youtube.com/watch?v=0DTYh37U8uE

19 Oct 2018: A summary and discussion of Turing's 1938 views on intuition and ingenuity in mathematics:
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/turing-intuition.html (also (pdf).

NOTE: Hatching mechanisms
Since late 2020 I have been discussing mechanisms required for controlling hatching processes in eggs of egg-laying vertebrate species, and the implications for evolution of some ancient mechanisms of spatial reasoning in those species. The ideas about hatching mechanisms and their evolution continued evolving during 2021 and 2022. A relatively recent version is available here:

https://www.cs.bham.ac.uk/research/projects/cogaff/misc/evo-devo-figs.html
Link inserted 11 Oct 2022

NOTE: Turing on continuously changing chemical processes
In 1952 Turing published a paper entitled "The Chemical Basis Of Morphogenesis", in *Philosophical Transactions of the Royal Society. London B*, pp. 37--72, https://doi.org/10.1098/rstb.1952.0012
The abstract states: "The purpose of this paper is to discuss a possible mechanism by which the genes of a zygote may determine the anatomical structure of the resulting organism". But the paper focuses only on 2D structure/pattern formation, which is clearly a relatively simple special sub-category of types of spatial structure formation.

I suspect that at that time he was working on a deeper, broader research project investing roles of chemistry in controlling assembly of 3D structures and also roles for chemistry in brain mechanisms for reasoning that are implemented in chemical mechanisms rather than in trainable neural networks that merely collect statistical evidence and derive probabilities.

The 1952 paper provided only examples from the 2D subset of structures/patterns whose formation could be explained by combinations of reaction and diffusion, but I suspect he was secretly working on a far more ambitious account of 3D structure formation, and perhaps related abilities to reason about more general forms of spatial structure formation.

Clearly, chemistry-based 2D pattern formation could not explain the formation of 3D chemical structures and mechanisms in brains and other parts of animal bodies. A much richer variety of chemical mechanisms is required in order to account for the formation of 3D chemical structures in living organisms.

A 2D structure cannot have a tube going all the way through it. So an organism implemented as a 2D structure cannot have an alimentary canal, that persists during a variety of spatial motions and deformations, whereas a 3D structure can.

This is a serious limitation of Conway's "Game of Life" which many researchers trying to use Conway's model to explain biological phenomena, seem not to have noticed. I suspect Turing must have been aware of the difference, but I have not come across evidence that he was.

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