



Reasoning About Continuous Deformation of Curves on a torus and other things.

(CHANGING DRAFT:

Stored copies will become out of date.)

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ABSTRACT

This is one of several online discussions, with examples, of kinds of mathematical competence that seem to have evolved from abilities to perceive and reason about proto-affordances[*] in the environment. In particular, humans, and possibly other organisms, can use structural relationships to generate a space of possible structures and processes, and then reason about constraints on those possibilities. This seems to require a kind of mathematical meta-cognitive (or meta-meta-cognitive) reasoning ability that may not be present in other species, and does not seem to be present in humans at birth.

This sort of mathematical discovery process is totally different from statistics-based discovery that uses a large number of observations to compute probabilities that support predictions made under conditions of uncertainty. The probability that everything with property P also has property Q can be high even if there are examples of P that are not Q, as long as they are relatively rare. In contrast we are here discussing mathematical discoveries that cannot have any exceptions.

Research on mechanisms for doing probabilistic learning and prediction now dominates a lot of research in cognitive science, neuroscience, AI, Robotics, and even philosophy. I suspect that's a passing fashion based on a failure to attend to the breadth of phenomena in perception, learning, and mathematical discovery.

Alternatives could come from the study of Euclidean geometry, but, unfortunately, our educational systems no longer teach all bright children to find constructions and prove theorems in Euclidean geometry. As a result, many highly educated researchers have never had the sorts of mathematical experience that are the subject matter of this paper, although they have had simpler mathematical experiences when they were toddlers, as illustrated here:

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/toddler-theorems.html>

Examples, some easy and some hard, are presented below. They require modes of perceiving and thinking that may have led to the production of Euclid's *Elements* over 2000 years ago. We still don't know how to model these mathematical discovery processes. Some suggestions are offered in terms of layers of meta-cognitive virtual machinery performing different functions. Other examples involving reasoning about triangles are presented in related documents, referenced below.

[] Most of J.J.Gibson's discussions of perception of affordances refer to affordances involving possibilities for and constraints on **actions that might be done by the perceiver**. Perceiving proto-affordances is more basic: it involves seeing possibilities for change in the environment no matter whether the perceiver or any other agent is involved in producing the change, or benefitting or suffering from the change. If Newton really did think about an apple he was thinking about proto-affordances involving the apple and other things including the tree and the ground below. The existence of proto-affordances does not depend on the existence of perceivers.*

This is part of the Meta-Morphogenesis project:

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/meta-morphogenesis.html>

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17 Apr 2022: more formatting changes and minor corrections.

16 Aug 2021: mainly formatting changes.

11 May 2015 (Moved most of the discussion about stone rings to: [a separate document](#)).

25 Apr 2015 (Added Fig 2 and question about stone rings)

20 Jan 2015 (Added note on construction kits);

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This paper is

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/torus.html>

A PDF version (possibly slightly out of date) can be found in:

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/torus.pdf>

This is closely related to discussions of functions of biological vision in <http://www.cs.bham.ac.uk/research/projects/cogaff/misc/vision> including the role of biological vision in human mathematical discovery, especially geometry and topology, e.g.

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/triangle-theorem.html>

A partial index of discussion notes is in

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/AREADME.html>

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How can you know certain types of deformation of a spatial structure are possible and others impossible?

NOTE:

I don't think there's a short answer to that (Kantian) question. My main purpose here is to present a variety of different types of example that are likely to be unfamiliar to readers, so that they are forced to think, and use their spatial reasoning powers, in order to have a variety of first-hand experiences to help drive some research questions, and also to provide a basis for rejecting shallow or otherwise inadequate explanations such as often come out of arm-chair theorising using only one or two well-worn examples, discussed in terms that don't generalise to other examples. This could be regarded as a sort of philosophical botany, helping to identify phenomena that need to be explained.

Compare Sloman ([2015](#)),

Returning to the question:

I'll start with a relatively easy example of geometrical reasoning involving shapes that can be formed by the overlap between a circle and a rectangle in a plane, and then move on to harder questions about possible continuous deformations of closed curves on the surface of a torus. In both cases we use an ability to consider possible variations of configurations in a space, subject to certain constraints (the circle and rectangle do not change their shapes, though they can be rotated and moved in the plane, the curves on the torus can move continuously subject to the motion being continuous without any self crossing).

In both cases we find that the specified set of possibilities necessarily satisfies additional constraints (certain things are impossible) and discovering this requires mathematical competences that are completely different from sampling a space of possibilities looking for regularities or probability distributions.

But some of the methods of reasoning that can be used, and are used by at least some humans, are also different from the kinds that use logical, algebraic and arithmetic forms of reasoning, based on abilities to manipulate and inspect logical and algebraic formulae, e.g. checking whether a sequence of formulae constitutes a proof in a specific formal system. That raises the question: what other forms of reasoning are available, or more precisely, what forms of representation, types of manipulation of representations and information processing architectures can account for those non-logical, yet mathematical, discovery and reasoning processes?

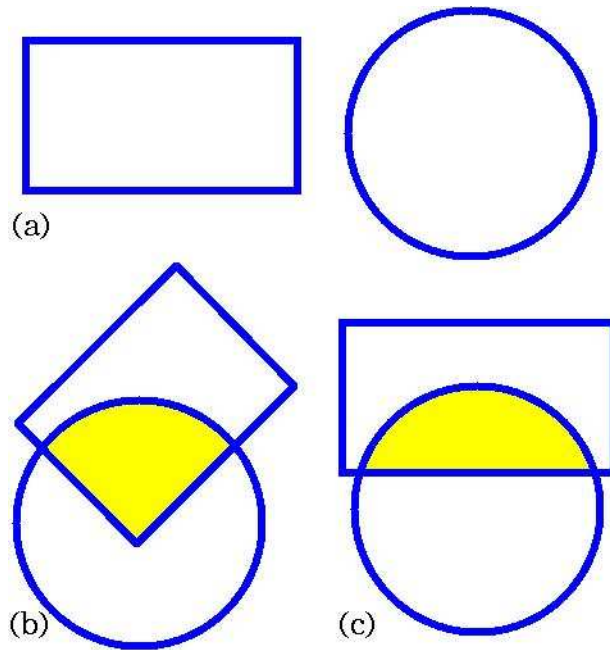


FIG 1 Configurations of a circle and a rectangle

Three configurations of a circle and a rectangle are shown above. In one configuration (a) there is no overlap between the circle and the rectangle. In the two lower configurations (b) and (c) they overlap, and the overlap area has a shape indicated in yellow. The two yellow shapes are different. For example, one of the yellow shapes has three vertices, and one has only two. One of them has a vertex where two straight lines meet (i.e. a corner) and the other has only vertices where a straight line meets a curve.

Can you imagine more configurations that could be produced by the overlap between a circle and a rectangle. How do you do that imagining?

How many different "overlap shapes" are there? It depends how you count them. If you consider how any two configurations, C1 and C2, (e.g. (a) and (b) or (b) and (c)), differ, you will be able to think of another configuration where one of the distances e.g. between a vertex of the rectangle and the circle, or between an edge of the rectangle and the centre of the circle is intermediate between the distances in C1 and C2. So there must be infinitely many possible configurations, though your visual system cannot discriminate them all. How can you be sure they exist? What do you have to know about space? (Compare Achilles and the Tortoise, in Zeno's paradox.)

Can you discover any limitations on the possible configurations? For example, suppose the rectangle is moved, while the circle remains fixed, so as to produce an overlap area whose boundary includes two corners, i.e. two vertices where straight lines meet? Is an overlap area possible with exactly four vertices where none of them are corners, i.e. all the vertices have a curved edge, unlike (b) where the bottom vertex is a corner, and unlike (c) where the yellow overlap region has only two vertices?

What's the maximum possible number of vertices that can exist in the area of overlap between a circle and a rectangle in the same plane, if there are no limits on the size of the circle or the size of the rectangle? How do you know those limits exist? Do you have to draw thousands of randomly generated configurations and inspect them? Or is there a better way to answer the question, by

analysing the process of generating all possible overlaps?

I suspect that one of the products of biological evolution was the mixed ability of some animals not only (i) to see things that actually exist in the space surrounding them but also (ii) to think about possible alternative configurations and processes, and in some cases (iii) to discover limitations on what's possible. This generalises some of the claims James Gibson made about the perception of affordances, discussed here: <http://www.cs.bham.ac.uk/research/projects/cogaff/talks/#talk93>

Impossible 3-D linking/unlinking

Here's an example involving two stone rings forming a chain on an old Indian temple: Vaideshwara temple at Talakad:



FIG 2 Stone rings

From Wikipedia

http://en.wikipedia.org/wiki/Group_of_temples_at_Talakad,_Karnataka

Picture by Hari, Ganesh R, subject to [Creative Commons Attribution-Share Alike 3.0 Unported license](http://creativecommons.org/licenses/by-sa/3.0/).

Could the two rings have been cut out of two separate blocks of stone and then assembled as shown in the image? This is discussed, along with a set of related questions about linking and unlinking rings or loops in a separate document:

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/rings.html>

Possibilities and constraints: Curves on a torus

The rest of this document raises questions of a similar kind, but in relation to closed curves on the surface of a torus. I don't believe any of the current models of vision, learning, or reasoning that have been proposed in psychology, neuroscience, AI or Robotics can account for the phenomena discussed here, even though there are many impressive graphical predictive programmes. A graphical program could start with one of the above configurations, then repeatedly modify it slightly, check for an overlap region and count the vertices. But however many times this was done it would not prove that the largest number found is the maximum. It would not even yield a

probability since any finite sample of configurations would be only 0% of the infinitely many possible configurations.

It is not clear yet what forms of computation would be required. Some brief suggestions are made below, but first a more complex set of questions will be presented, involving closed curves on the surface of a Torus.

Consider this (artistically not very good) picture of a torus, with some curves on its surface: **FIG 3**

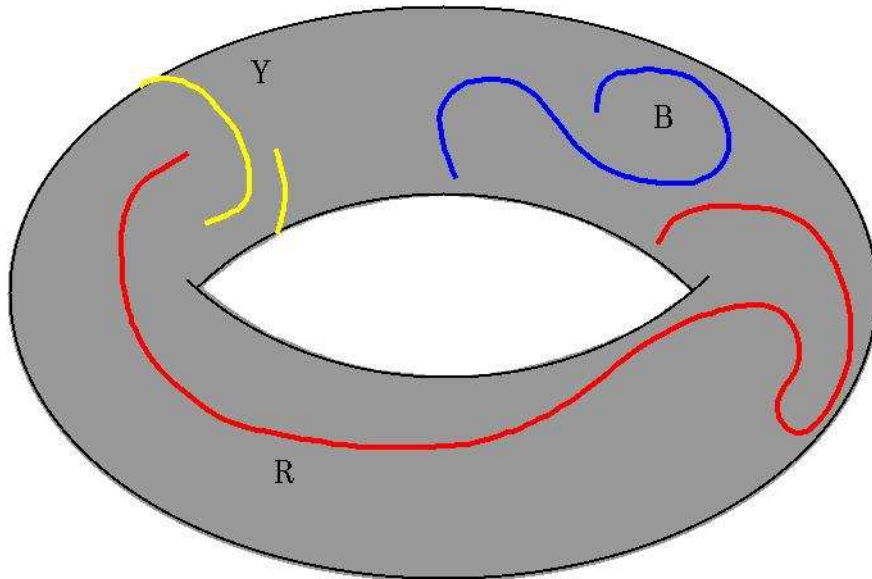


FIG 3 Curves on a torus

You should have no difficulty interpreting that as a picture of a 3-D torus-shaped structure (a closed tube). Three curves are drawn on the surface, one Red, one Blue one Yellow (partly hidden by the torus). It is not hard to think about the curves being moved continuously in the surface, so that no breaks occur and no curve ever crosses itself. Is it possible to deform the blue curve B by sliding it continuously in the surface so that it ends up lying exactly where the red curve, R, is?

You can also see, and may have noticed without being told by me, that it is possible to move B to where R is in different ways, e.g. depending on which end of B goes to which end of R, and whether any part of B moves over any part of R before the final slide into place.

Likewise curve B could be moved continuously into the location occupied by curve Y (partly out of sight), or Y could be deformed to lie on the location of B or R.

All the curves are in the same "equivalence class": each can be continuously deformed into all the others. This is also true of simple, non-closed, non-self-crossing continuous curves on a plane, on a sphere, or on an ellipsoid.

Now consider **FIG 4**, below, containing five curves, B1, B2 both blue, R1 red, and Y1, Y2 yellow. This time all are closed curves: they have no free ends.

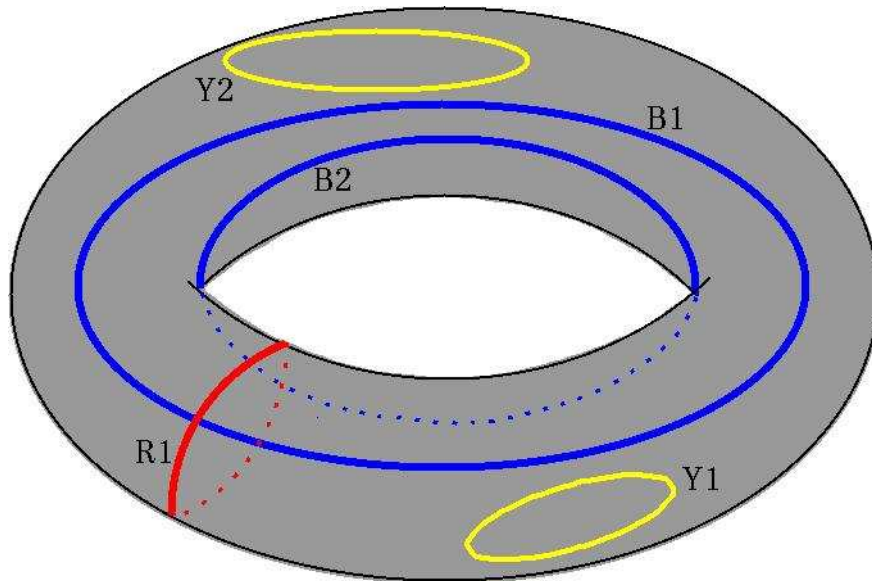


FIG 4 Closed curves on a torus

You can probably tell that the two yellow closed curves Y1 and Y2 are mutually continuously transformable: each can be smoothly moved in the surface of the torus to occupy the exact location of the other. How do you convince yourself that it is possible? Do you have to physically create a succession of intermediate curves, or is it enough to imagine them? Do you have to imagine all of the intermediate locations, and is the equivalence obvious at a more abstract level?

Some stretching and alteration of curvature (e.g. the sorts of things you can do with an elastic band) may be required, but no cutting and joining, and no self-crossing (as in a figure eight: "8").

Likewise the two blue closed curves are mutually continuously transformable, B1 wholly visible in the picture and B2 continuing out of sight on the underside of the torus as shown by the dotted continuation, though neither is mutually continuously transformable with either Y1 or Y2. Why not? How can you be sure?

Moreover, none of the blue or yellow curves can be moved continuously in the surface of the torus to occupy the location of the closed red curve R1, which continues out of sight around the torus to complete the loop as shown by the dotted portion.)

These examples illustrate the fact that besides what exists in the environment, we can also perceive and think about ways in which what exists could have been different or could change in future, and we can also discover cases that cannot exist. In other words we can discover **possibilities** and **impossibilities**, and, for humans these (essentially mathematical) capabilities are not particularly unusual, although some of the examples may be too difficult for a large subset of the human population.

These discoveries seem to be closely related both to abilities of humans and other animals to perceive, reason about, and make use of *affordances* of various sorts in the environment, and to abilities to make mathematical discoveries and to reason about them.

Such capabilities must have evolved and been used by humans, then used in the production of Euclidean geometry long before the development of axiomatic, algebraic, or logic based mathematics, and before Descartes discovered the mapping of Geometry into Arithmetic. Euclid's

Elements was produced around 25 centuries ago.

Lifting a mug with a pencil

FIG 5, below, shows two 3-D objects that you can probably recognise as a mug and a pencil (though some of your ancestors might not have found that so easy). The picture also includes partial views of some additional objects and surfaces in the background, which you may ignore except for the horizontal surface supporting the mug.

As with the previous examples you are able to look at a particular configuration and imagine ways in which the relationships could have been different. The mug might have been lying on its side, or its handle might have been jutting out in a different direction. The pencil could have been resting on the top but shifted slightly to one or other side, or rotated to point in a different direction.



FIG 5 Pencil on mug

A task: if you had to use one hand to lift the mug without touching it (e.g. so as not to leave finger-prints on the surface) could you use the pencil to do so? If you had a real mug and pencil you could pick up the pencil, holding it in a wide variety of different ways, and for each way of holding it try a wide (infinite) variety of random movements to find a sequence of actions that cause the mug to be lifted upwards off the surface supporting it. However, without performing any movements, let alone an infinite variety, I suspect you can think of at least one way, and possibly

more than one distinct way, of achieving the lifting, using the pencil to transfer appropriate forces to the mug, even if you have never previously been faced with this challenge (so that your solution will be an example of P-creativity, personal creativity, in Margaret Boden's sense).

NOTE

I have deliberately not described possible solutions, i.e. possible ways of moving the pencil to lift the mug, as I think it best for readers to find them without help, and then reflect on what they have done and how they did it and what capabilities and mechanisms (forms of representation, modes of reasoning, algorithms, information processing architectures, ...) a machine might need to support those discovery processes. I'll return to curves on a torus, below.

Humans (though not newborn infants) and many other intelligent animals seem to be able to look at a 3-D configuration of objects and to reason about possible changes of relationships that will enable some desired end state to be achieved. Since the space of possible spatial changes is continuous the variety of possible changes is multiply infinite (infinitely many initial possible changes each followed by infinitely many further possibilities). Searching that space exhaustively either by actions or using imagined actions is impossible.

One way to tame the infinitude is to chunk possible changes into different subsets that share some common properties, and then explore each subset by considering a particular specimen at a high level of abstraction. Humans and some other intelligent animals seem to be able to do this, but without first producing then manipulating logical or algebraic descriptions of the relationships, which, as far as I know, no non-human animal can do and most of our ancestors could not do, since logic and algebra are relatively recent human discoveries.

Your ability to think of possible ways of moving the pencil subject to various constraints implied by the position and shape of the mug is similar in important ways to your ability to think about possible configurations of a circle and rectangle, or possible configurations of curves on a torus. I suspect the kind of mathematical knowledge that we now think of as geometrical or topological knowledge may have resulted from reflection on features of our abilities to imagine and reason about possible changes of spatial configurations subject to constraints, which other animals can also do, but without noticing that they do it -- for instance a weaver bird making a nest hanging from a branch, using hundreds of long thin leaves knotted together, demonstrated in this video:

<https://www.youtube.com/watch?v=6svAlgEnFvw>

As far as I know, no current robot can do this exploration, in imagination, of continuously varying sets of possibilities, including dividing the possibilities up into subsets sharing common constraints, each generating new sets of possibilities. And neuroscientists don't know how brains do it (if they have noticed that this is something brains can do -- which I suspect is not something most neuroscientists or cognitive scientists have noticed!).

Studying how such capabilities develop in young children would require Piagetian longitudinal studies of individual development, rather than the current shallow statistics-collecting experiments that are misleadingly labelled "science" in many laboratories. But empirical research will not be enough (as Piaget understood). We'll need some deep new theories that can be tested in working robots.

Currently fashionable theories about brains as statistical predictors cannot even begin to explain these things, since those theories deal with **probabilities**. They avoid the deeper problems of representing **possibilities** by assuming that everything can be represented as collections of numerical measures, which naturally generate sets of possibilities -- the wrong sets for the kinds of reasoning discussed here -- a practice criticised in [Sloman \(2007\)](#).

Research challenges

As far as I know, nobody has been able to model these mathematical reasoning capabilities in computers, despite significant advances in logical, arithmetical and algebraic theorem-proving, with many applications. Further, I don't even know of researchers who have understood the *requirements* for robot vision that are connected with human-like mathematical discovery processes. Robot research, and much vision research in AI and psychology omits important requirements analysis, so the theories and models produced are tested only against much shallower challenges.

Relevance to philosophy and possible-worlds semantics

Many philosophers (REF) believe that notions of *possibility* and *necessity* can be understood only in terms of what is true in some possible world (possible complete universe) and what is true in all possible worlds (universes). But it is clear that the examples of possibilities and impossibilities presented above can be discovered and thought about without any need to consider the whole state of the universe (this world) nor alternative possible states of the whole universe (possible worlds). A complete refutation of the alleged explanatory power of possible worlds semantics will have to wait for another occasion.

Equivalence classes of closed curves

Two closed, non-self-crossing, curves on the surface can be described as being in the same *continuous deformation equivalence class* if either curve can be continuously deformed on the surface of the torus (i.e. without passing through any part of the interior of the torus, and without being lifted off the torus) into the other, without breaking and rejoining and without crossing over itself at any point (as a figure of 8 does), and with no two portions of the curve superimposed, e.g. by squashing a curve into a line.

NOTE (Added 3 Jul 2014):

The addition of the "no portions superimposed" condition was a result of a suggestion by Sebastian Zurek that one of the yellow curves (e.g. Y1) could be deformed continuously into one of the blue curves -- by first continuously squashing the yellow curve to form a line, then moving the two ends of the line around the hole in the torus, to meet on the far side, forming a curve continuously deformable into B1 or B2. Since I had not thought of that sort of possibility before talking to him I could not say that his proposed deformation was explicitly ruled out by the original description of permitted continuous deformations, though if the "no cutting and joining" condition is understood as "no cutting and no joining" (rather than "no cutting and rejoining of cut ends") then his proposal is ruled out.

This is an example of a not uncommon occurrence in mathematical discovery: a formulation of a generalisation that was thought to be adequate for specifying a mathematical truth turns out to allow an unthought of possibility as a counter-example, requiring the formulation of the generalisation to be made more precise, to rule out the counter-example. As noted by Lakatos

(1976) this implies that mathematical discovery and reasoning can be regarded as "quasi-empirical" because many sub-cases may have to be explored.

See: Imre Lakatos,

Proofs and Refutations, Cambridge University Press, 1976.

An infinity of curves

There are three classes of curves illustrated in the picture, the yellow curves, the blue curves and the curves equivalent to the red curve. The picture shows only two yellow curves two blue curves and one red curve, but each class has infinitely many members. How can you know that?

Compare: how can you know that there are infinitely many points, or line segments, on a straight line, even if it is a short straight line?

How can you be really sure that the three classes shown are not in the same larger class, i.e. how can you be sure that no curve in one of the three classes can be continuously deformed into any curve in either of the other two classes?

Is it possible that making a torus of the right material will allow the blue curve to be transformed continuously into either one of the yellow or red curves (without changing colour)?

Could the presence of a very strong magnetic field, nearby or on distant planet make a difference to any of these questions? If not, how can you be sure? This is similar to questions that can be asked about proofs of theorems about areas or angles of triangles in:

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/triangle-theorem.html>

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/triangle-sum.html>

Are there any other equivalence classes?

Can you think of any other continuous, closed, non-self-crossing curves that could exist in the surface of the torus, and are not in the same equivalence class as either the yellow, the blue, or the red curves? How many different equivalence class can you distinguish? How do you know they all exist on any torus?

You might consider allowing the torus itself to be deformed, e.g. shrinking, growing, bending round to touch itself, and other possibilities. Do any of them change the variety of equivalence classes of closed curves on the surface? How do you know? What brain mechanisms make it possible for humans to think about and find answers to such questions?

E.g. what happens to the various sorts of closed curve discussed so far if the "tube" comprising the torus increases its cross section, so that the hole in the middle gets smaller and smaller, and then becomes a single point?

Note (6 Jul 2014):

When I wrote an early version of the above discussion, I was not aware that there are infinitely many distinct equivalence classes of closed curves on the surface of a torus. (Bob Durrant originally drew my attention to the existence of two distinct infinite families of equivalence classes of closed curves on a torus.)

Are there only two such classes, or are there additional classes of closed, continuous, non-self-crossing curves?

If you have found any such curves, or sets of curves, not mentioned here I would appreciate being informed by email, with a note of how you found them, if possible and a reference to a relevant publication (preferably freely available online).

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What about curves in a plane surface or on a sphere?

How much of what's presented here in connection with curves on a torus transfers to curves on a sphere, or curves on an infinite flat surface?

Square and Torus (Added 6 Jul 2014)

A square, or a rectangle, can be converted to a torus by first joining one pair of opposite edges, to make a tube, then joining the ends of the tube, now occupied by the other pair of opposite edges of the original square. What follows from the fact that it doesn't matter which pair of opposite edges you join first: both orders of joining will yield a torus. (Do you find that obvious?) What follows about the relationships between curves on a plane surface and curves on a torus?

(I thank Achim Jung and Bob Durrant for reminding me of that relationship, familiar to many designers of graphical interfaces for games or moving pictures.)

What differences can you make to torus equivalence classes with a knife?

(Added 30 Jun 2014)

There's a collection of questions about a torus whose answers are already familiar to people who have experience of interacting with bagels and doughnuts, including questions about what can be changed by slicing through a torus in various ways, using a planar cutter.

Consider a knife of finite length. If the knife is long enough, a single slice, moving the blade along a plane, will change the doughnut by dividing it into two separate pieces. Try to imagine the various ways you can slice a doughnut into two (or more?) separate parts using a single slicing action, and what effects they will have on the blue, yellow and red curves shown above.

Assume the knife is long enough to protrude beyond the torus (doughnut) in both directions along its length, and it is always moved within a fixed plane, while the torus is held fixed relative to the plane.

It is also possible to slice right through a doughnut without producing two separate parts. How would such a slice affect the different sorts of curves?

Many will be tempted to think that the answers to such questions are empirical, and depend on individuals having experience of slicing doughnuts. But a mathematician will recognize the questions as examples of non-empirical mathematical thinking of a type that can lead to advances in mathematics that may be applicable to the physical sciences and to engineering, but are not generalisations from examples that can be refuted by repeating the same experiment many times. That's because the knowledge acquired is about sets of possibilities and constraints on those possibilities, rather than about probabilities summarising ratios of observed instances of those possibilities.

Conjecture: affordances and the origins of Euclid's elements

Questions of the type raised in this note can be understood by intelligent human beings who have never heard of logic, the axiomatic method, or cartesian coordinate representations of geometry, a discovery made only a few centuries ago by Descartes and unknown to the great thinkers who produced Euclid's *Elements*.

Not everyone will understand all the questions, and not everyone will be able to answer the questions posed here. Some are too young, with underdeveloped brains. Some may be old enough but have not yet learnt to think mathematically. Some may be incapable of acquiring that sort of ability, or may be able to do it for simple sorts of mathematics but not the most complex and abstruse sorts. (I have great difficulty understanding some mathematical presentations of advances since I was a mathematics student half a century ago.)

But those points all illustrate the well known fact that human minds are not all alike, for various different reasons that are not yet well understood. E.g. it's not clear that all the differences in reasoning capabilities are due to environmental influences, any more than differences in height, eye-colour, or possession of a neuro-developmental abnormality such as Down syndrome can be explained by environmental influences on the individual.

It seems that biological evolution produced organisms that are capable of various kinds of mathematical reasoning, though only humans seem to be able to discover that they have those capabilities and to discuss the abilities and help other individuals to acquire them. And many are acquired without being recognised, discussed or thought about, e.g. in very young children.

Toddler Theorems

That idea is developed in the discussion of 'Toddler' theorems here:

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/toddler-theorems.html>

That is part of the Meta-Morphogenesis project, discussed here:

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/meta-morphogenesis.html>

There's more to be said about the role of mathematics in biological evolution, long before human mathematicians or most of their ancestors existed, as illustrated here:

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/bio-math-phil.html>

Examples concerned with triangles can be found in these two files:

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/triangle-theorem.html>

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/triangle-sum.html>

Affordances: Gibson and Euclid

I suspect there are many more aspects of our ability to reason about structures and processes that are closely connected with the ability to perceive spatial structures and processes and to understand the affordances they provide -- either for the perceiver or for others. (The notion of "affordance" used here is a generalisation of J.J.Gibson's ideas.) Some of the requirements for vision systems are discussed in:

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/vision>

Some of these discoveries seem to depend crucially on the ability of a reasoner to inspect processes of reasoning -- as they occur or retrospectively -- and to discover features that are not necessarily noticed by all individuals that use those reasoning processes. I suspect this is possible only in machines running sophisticated virtual machine architectures with properties that human engineers did not find useful until the late twentieth century -- and are still not fully understood in some cases. I'll enlarge on the architectural requirements below. Some aspects of virtual-machine functionalism relevant to this are explained here:

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/vm-functionalism.html>

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Could new forms of computation (information processing) help?

As far as I know, no current AI or Robot vision system has the sorts of abilities illustrated here, and no AI vision researchers or mathematical reasoning researchers are attempting to give robots these capabilities.

It is not clear whether the currently understood computational mechanisms, information-processing architectures, forms of representation or forms of reasoning can provide the required explanations. That's connected with the possibility that they may be unable to explain the kinds of mathematical discovery and reasoning processes that led up to the production of Euclid's Elements, thousands of years before the development of modern mathematics and modern logic. They must have been using mechanisms we have not yet understood, or using mechanisms familiar to us but using them in an unfamiliar way.

Irrelevance of statistical/probabilistic learning and reasoning

None of this has anything to do with discovering statistical properties of observations or sensory-motor signals. The mathematical discoveries alluded to here and in the linked papers are not concerned with probabilities, but with sets of possibilities and constraints on those possibilities that give rise to entailments and impossibilities.

Immanuel Kant

I believe the discussion above consistent with the philosophy of mathematics developed, though in a rudimentary form, by Immanuel Kant, in his *Critique of Pure Reason* (1781).

There is also a deep connection between his philosophy of mathematics and his ideas about causation, because there are some mathematical relations that are causal. E.g. adding two marbles to a cup containing exactly six marbles causes the number of marbles in the cup to go up to eight. Moving a vertex of a triangle further away from the opposite side of the triangle without altering the length of that side, causes the area of the triangle to increase. Altering the curvature of a line in a plane causes infinitely many straight-line distances between points on the line to change -- though the exact formulation of those changes depends on how points on the line are identified during continuous deformations (including stretching).

Kant, if I've understood him, noticed that for an intelligent individual to develop in a rich and complex spatially embedded environment and to make discoveries about what sorts of things exist in the environment, what they can do, how they interact, etc. it needs to have some assumptions about types of necessity, including causal necessity, as well as types of possibility in the environment. But that's a topic for another occasion.

The books and papers by Max Wertheimer, on productive thinking, John Holt, on how children learn, Piaget on many topics in child development, are all relevant.

There is a vast amount of research in developmental psychology on what infants and children can do when, and often attempts to find out why they fail to be able to do things. But the researchers mostly have no idea how anyone does anything, and they could not build working robots to show how the relevant perceptual, learning, and reasoning processes actually work. The same can be said of neuroscientists, and all the researchers who think perception and learning are always inherently statistical/probabilistic. They have either never made even elementary discoveries in geometry, topology, logic or arithmetic, or perhaps simply failed to notice how unlike probabilistic reasoning mathematical discovery is.

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This is part of the Meta-Morphogenesis project

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/meta-morphogenesis.html>

The meta-morphogenesis (M-M) project, was first introduced in

Aaron Sloman, 2013,

Virtual Machinery and Evolution of Mind (Part 3)

Meta-Morphogenesis: Evolution of Information-Processing Machinery,
Alan Turing - His Work and Impact,

Eds. S. B. Cooper and J. van Leeuwen, Elsevier, Amsterdam, pp. 849-856,

<http://www.cs.bham.ac.uk/research/projects/cogaff/11.html#1106d>

It is relevant to discovery and explanation of the role of mathematics in various evolutionary developments, as well as (later) cultural roles.

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/bio-math-phil.html>

NB: The claim that mathematical discoveries are non-empirical is not intended to imply that mathematical knowledge is innate (Kant made that distinction very, clear) and is not intended to imply that mathematical thinking is infallible.

Anyone who has done some mathematical thinking knows that mistakes can be made, and can sometimes be hard to discover. *Proofs and Refutations*, by Imre Lakatos (1976), presents examples in the work of some great mathematicians. He used the label 'quasi-empirical' to characterise mathematical discovery, which I take to refer to the exploration of complex mathematical structures and spaces of possibilities, which has partial similarities with, but should not be confused with, empirical exploration in physics, chemistry, geology, biology, psychology, sociology, etc. One similarity is that different parts of a complex structure or space come into view during empirical exploration and during mathematical exploration. What is different is the nature of what can be discovered.

NOTE: Added 30 Jun 2014

Kant also claimed that the necessary truths of mathematics (including arithmetic and geometry) are synthetic, i.e. not analytic -- or, in Hume's terms, mere "Relations of ideas", by which Hume seemed to imply that they were essentially trivial (e.g. definitional) truths that may extend our terminology but not our knowledge about anything. The concept of analyticity was sharpened by Frege, who attempted to specify in detail the nature of logic and what could be proved using logic. He thought (contra Kant) that arithmetical knowledge was analytic (derivable from definitions using purely logical reasoning), but agreed with Kant that geometrical knowledge was not analytic, i.e. was synthetic. I presume that he would have included the kinds of topological knowledge presented above. Any reader who claims to be able to demonstrate any of the claims about equivalence or non-equivalence of closed curves on a torus using only logic and definitions, should contact me, as I would be interested to see the definitions and the proof. My guess is that any such proof would not be comprehended by most of the people who grasp the mathematical possibilities and impossibilities presented here. If anyone claims that we are unconsciously using nothing but logic, it would be very interesting to see the evidence, including an explanation of how a computer-based robot might be designed to make these discoveries using only logical reasoning.

I have not yet attempted to find out whether the claims about curves on a torus could be proved using one of the axiomatisations of Euclidean geometry, e.g. by Hilbert.

For more on the distinctions used here see

A. Sloman (1965),

"Necessary", "A Priori" and "Analytic", *Analysis*, 26, 1, pp. 12--16,

<http://www.cs.bham.ac.uk/research/projects/cogaff/07.html#701>

[ADDITIONAL REFERENCES TO BE ADDED]

FURTHER THOUGHTS ON MECHANISMS REQUIRED**[Still under construction]****(Added 7 Jul 2014)**

After some discussion with Achim Jung and other colleagues in the Theoretical Computer Science group at Birmingham University, who did not know of any automated reasoning system capable of making the discoveries about curves on a torus, I added this section.

Comments by Achim Jung are preceded by '>'

One of the facts about the discoveries concerning a torus, and discoveries about triangles, circles, etc. in Euclidean geometry is that some of them are easier for humans than others. E.g. the equivalence of the yellow curves seems to be easier to see than the equivalence of the blue curves.

Likewise some cases of reasoning about knots (e.g. whether pulling two ends of a piece of string apart will produce a knot or a straight string) are also difficult, and others easy. If we can find ways of automating the easier cases we may later find mechanisms for generalising them.

Similar comments can be made about theorems involving continuous deformation of polygons, e.g. triangles:

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/triangle-theorem.html>

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/triangle-sum.html>

However, the restrictions to straight lines make it easier to *replace* those problems by *structurally equivalent* problems in algebra, using Descartes' discoveries.

Whether the algebraic proofs can be regarded as proofs of the original Euclidean discoveries depends on whether there is a proof of the equivalence of the spatial domain and its image in the numerical domain. I suspect any such proof will have some of the same difficulties as the original proofs using geometric or topological reasoning.

[AJ]

> ...in some sense, algebraization is the attempt of mathematicians
> to get a grip on them (quite successful for curves on a torus:
> algebraic topology and group theory; still difficult for knots).

Yes, but we also need to explain the kind of mathematical reasoning that enables the **adequacy** of the algebraization to be recognised. That isn't done by the algebraization. It requires deep reasoning about the relationships between the original domain and the arithmetized domain. (Many mathematical advances are concerned with relationships between different domains of structures and transformations of structures, i.e. processes.)

[AJ]

> Both curves on a torus and knots are quite hard; I wonder whether
> you have looked at the problem (impossibility) of continuously
> deforming a curve in the plane across a hole.

or even across a single missing point (infinitely small hole). Yes. That came up as an example in a philosophy seminar in January. I had not previously encountered the example.

[AJ]

> (By the way, the study of curves on a torus can be reduced to
> curves in the plane.)

Yes: a square in a plane can have opposite edges 'wrapped' to form a torus.

[AJ]

> It seems to me that for humans the intuition stems from the
> (typically very rich) physical experience of moving objects.
> Whether we humans do more than just pattern match in a
> four-dimensional space (space + time) I don't know.

Exactly how the intuitions "stem" is in part what we need to explain. It cannot merely be the discovery of statistical correlations, since such discoveries cannot provide knowledge of impossibility or necessity, both of which are absolutes (opposites of each other), whereas statistical correlations vary in degree.

There's a vast amount of energy now going into designing (and talking about) systems that do statistical (e.g. Bayesian) learning, while ignoring the vast body of (more or less ancient) types of mathematical learning and engineering design, based on non-statistical (and not purely logical) mathematical understanding (including software engineering design) that has nothing to do with learnt probabilities.

That includes the discoveries reported in in Euclid's *Elements*.

[AJ]

> "Reasoning" of any kind seems to require some clever form of
> representation and abstraction.
> (Hilbert's "Grundlagen der Geometrie").

Yes. However, Frege (in his *Grundlagen der Arithmetik*) argued that Hilbert's claim to be talking about Euclid's geometry was not justified.

If I have understood him correctly, he essentially said that Hilbert had identified another domain in which he could prove theorems, but could not prove the equivalence of that new domain and the original geometry within his system.

In contrast, Frege (like Russell and Whitehead) argued explicitly that in his logicization of arithmetic he had captured the *original* arithmetical domain.

However, I think the same arguments (or similar arguments) can be used against Frege as he used against Hilbert, because the currently assumed domain of arithmetic, e.g. based either on Frege's definitions or a Peano-based axiomatisation, is equivalent to a domain of cardinals that had previously been discovered without using any of Frege's logical apparatus (although the ancient concepts of cardinal number depend on a sort of topological notion of one-one correspondence), and the logical systems used as a basis for arithmetic cannot be used to prove that structural equivalence between the ancient familiar domains of cardinal or ordinal number and the newly specified logic-based domain/domains (or perhaps partial equivalence, depending on how you view the relationship between the geometrical continuum and the Cantor continuum).

A deeper, more general, type of mathematical competence is required to establish the equivalence, in both cases (old and new geometry, old and new arithmetic).

I am trying to identify that deeper mathematical competence, which was originally produced by biological evolution, driven by the structures of the environments in which organisms evolved, plus perhaps social interactions between our ancestors.

I don't know whether humans will ever understand the brain mechanisms involved. Kant seemed to think not, but perhaps he was too pessimistic.

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Levels of Virtual Machinery required

I think that ancient mathematical competence requires (at least) three, and possibly four, levels of virtual machinery (not all present at birth in humans):

1. mechanisms for modelling structures, relations between structures, and transformations of structures, found in the environment.
2. mechanisms for noticing regularities (possibilities, impossibilities, and invariants in such structures and processes).
3. mechanisms that can do meta-reasoning about the first two mechanisms in order to show that the discoveries in 2, are not "mere" empirical discoveries about what happens in our environment.

(Such empirical regularities -- like UP = OFF for electric light switches -- would not generalise to another continent, or a mountain top, or a very deep hole, or another planet, or very high temperatures, etc. etc...)

4. an additional layer of mechanism required for engaging in discussions and debates with other individuals about the discoveries, e.g. asking challenging questions, offering proofs, finding flaws in proofs, amending proofs or theorems, etc. etc. (I.e. all the stuff documented by Imre Lakatos in *Proofs and Refutations*).

Type 4. requires the ability to represent differences between what you know or believe and what another individual knows or believes, and that seems to go beyond type 3. However, it may be that even type 3 requires the ability to argue with *yourself*, using mechanisms of type 4.

I think these mechanisms start developing at different stages in human children.

Other species may have levels 1 and 2, or in some cases levels 1, 2, and 3 (e.g. corvids, weaver birds, elephants, some apes, etc.)

There may be more intermediate mechanisms than these.

Current AI systems that interact with physical environments (e.g. all 'autonomous' robots) seem to me to be restricted to very narrow subsets of levels 1 and 2.

AI theorem provers have something like levels 2 and 3, but restricted to an abstract environment composed of sequences of discrete symbols, and sequences of sequences, etc. Roughly, they are derived from post-Descartes mathematics by throwing away all the stuff in Euclid and the precursors used in coping with physical environments.

Maybe next year I'll work out how to build the required mechanisms. Or maybe next century -- or perhaps later. I suspect the hypothesised construction of "new levels" is closely related to Annette Karmiloff-Smith's ideas about "Representational Redescription" discussed in [a partial review](#) of her 1992 book

Beyond Modularity, and also related to the Meta-Configured genome idea:

<http://www.cs.bham.ac.uk/research/projects/cogaff/07.html#717>

Jackie Chappell and Aaron Sloman, 2007,
Natural and artificial meta-configured altricial
information-processing systems,
in *International Journal of Unconventional Computing*,
Vol 3, No 3, pp. 211--239,

Later followed by the ideas here:

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/meta-configured-genome.html>

Turing wrote, in his Mind 1950 paper:

In the nervous system chemical phenomena are at least as
important as electrical.

But he did not say why. Two years later his paper on the chemical basis of morphogenesis was published. I suspect he saw important connections that are now waiting to be re-discovered.

Further discussion

Olaf Klinke made some useful comments on an earlier version of the above and gave me permission to make them available here, with my responses:
<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/torus-olaf-klinke.html>

Chaining Rubber Bands problem (only loosely related)

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/rubber-bands.html>
Some possibilities and impossibilities involving linked rubber bands.

Added: 10 Jul 2017:

Just found: An entertaining discussion, begun in 2006:

<http://www.mystcommunity.com/board/index.php?/topic/23182-interlocking-two-rubber-bands/>
Torus, mobius strips, bagels, breakfast...
Alas this seems to be no longer accessible (18 Apr 2022).

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FURTHER READING (Still very messy: to be revised.)

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Aspects of the theory of syntax,
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