Surprisingly Rational Circle Segments
(DRAFT: Liable to change)

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This paper is
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/multicirc.html
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/multicirc.pdf

A partial index of discussion notes in this directory is in
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/AREADME.html

Introduction

I'll state a problem, and present a solution, below. Some readers may prefer to confront the problem without any hints regarding a solution. There is a separate web page that states this problem without presenting a solution here:
multicirc-problem.html http://www.cs.bham.ac.uk/research/projects/cogaff/misc/multicirc-problem.html

It makes some comments about the nature of the problem and its relationship to Immanuel Kant's ideas about mathematics, and mathematical limitations of AI systems based on statistical/probabilistic reasoning, e.g. Deep Learning mechanisms.

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Touching Circles

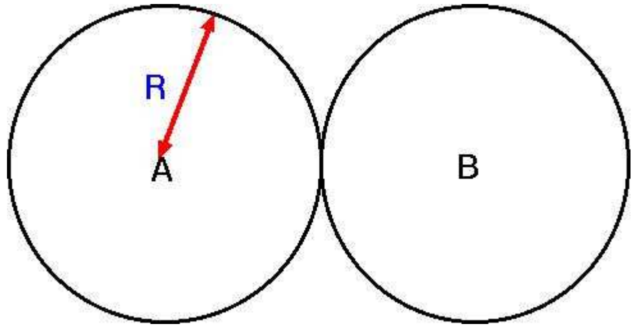


Fig 1 Two circles with centres at **A** and **B** of radius **R** touching as shown.
If two circles touch at a point (i.e. each is tangent to the other at that point) then their centres and that point are co-linear. How do you know that's true?

I'll use two touching circles, and a third circle passing through the point of contact, to formulate a problem, then show how to solve the problem, in a surprising way, by embedding the three circles in a larger, more complex four-circle structure, in which some relationships become "obvious", thereby revealing the solution to the problem.[Thanks]

Initial problem statement

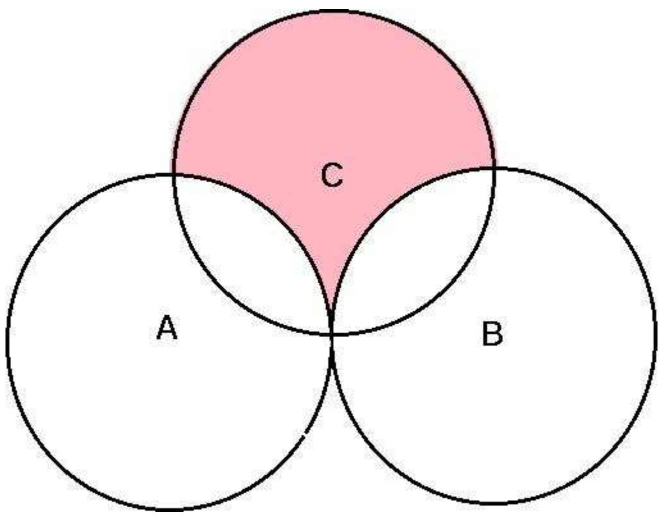


Fig 2 Add a third circle with centre **C**, also of radius **R**, above the line through **A** and **B**, with **C** placed symmetrically in relation to circles **A** and **B**, and passing through the point of contact of circles **A** and **B**, as shown. The line **AB** must then be a tangent to **C**. Why?

Note: The symmetry specified in Fig 2 implies that the centre of circle C is perpendicularly above the line joining the centres of the circles A and B. The centre of each circle must be a distance **R** from the intersection point, since they all have the same radius: **R**.

QUESTION
What is the area of the portion of circle **C** in Fig 2 that is outside the circles **A** and **B**, i.e. the area of the darker region?

That looks like a difficult question to answer because of the peculiar shape of the darker region. It is bounded by a convex curved portion at the top of circle **C**, and two concave portions below, meeting at a pointed cusp, where circles **A**, **B** and **C** intersect.

We'll adopt a round-about strategy for working out the area, using a new drawing, Fig 3, below, which has a new circle **D**, also of radius R, embedded in it, but without the shading in Fig 2.

Circle D is placed below Circle C, symmetrically in relation to circles A and B, and passing through the point of contact of circles A, B, and C as shown in Fig 3.

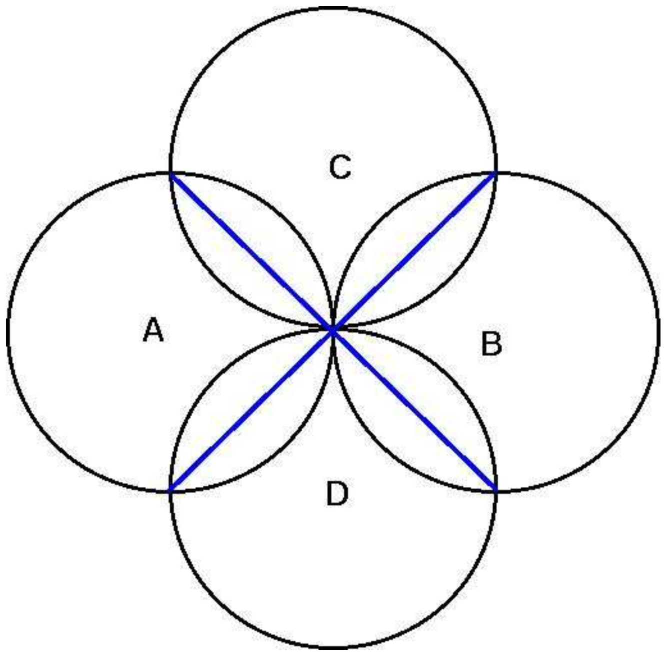


Fig 3 There are several ways of thinking about this figure, derived from Fig 2.

The original two circles **A** and **B** and the two new circles **C** and **D** interact to produce new intersection points, in addition to the intersection point at the center where they all meet. In Fig 3 blue lines have been added joining intersection points common to pairs of circles.

Because the construction has vertical, horizontal and diagonal symmetries, the four short blue lines form two co-linear pairs, forming two longer blue lines intersecting at the point where the circles intersect.

By reasoning about features of the new figure we'll find a simple way to calculate the shaded area in Fig 2. (You may already find it obvious.)

Note on creation of Fig 3 from Fig 2

One way to create Fig 3 from Fig 2 is to produce Circle **D** and the lower two blue lines by reflecting region **C** through the horizontal line joining **A** and **B** in Fig 2.

Another way is to construct Circle **D** in Fig 3 in the same way as Circle **C** was constructed in Fig 2: i.e. draw a new circle of radius **R**, with center at distance **R** perpendicularly below, not above, the point of contact of circles **A** and **B**, then add the lower two blue lines joining points of intersection of new and old circles. Here the process of construction is the "mirror image" of the process of construction of Fig 2.

Whichever way Fig 3 is created, it now has four symmetrically located regions similar to the shaded region in Fig 2, each with a sharp cusp pointing at the centre of the whole figure, where all four circles meet.

If L is the length of each blue line in Fig 3, then either of the above constructions produces four blue lines of the same length L, each joining a pair of intersection points of two overlapping circles.

The blue line segment of length L passing between A and D, and the blue line segment of length L between C and B must be collinear and of equal length. (Why?)

Likewise the two blue line segments passing between A and C, and between D and B must be collinear and of equal length L.

Consequently, there is a pair of blue lines, each of length 2xL, intersecting at right angles, and at their midpoints, in the center of Fig 3, where all the circles intersect.

Fig 2 was constructed to be symmetric about a vertical line. So reflecting it to make it symmetric about a horizontal line, produces a new figure with both vertical and horizontal symmetry. We'll use this as the basis for constructing the next figure, with horizontal and vertical lines through the centre of the figure, shown as dashed black lines in Fig 4, below.

We then add red lines joining pairs of end points of the dashed lines. Because of the symmetries in the construction process, the red lines form a square as shown in Fig 4.

There are now eight circle sectors outside the red square, but inside the circles, in addition to the eight sectors inside the red square, previously visible as areas of overlap in Fig 3, above, where they were separated by the blue lines, also shown in Fig 4, below.

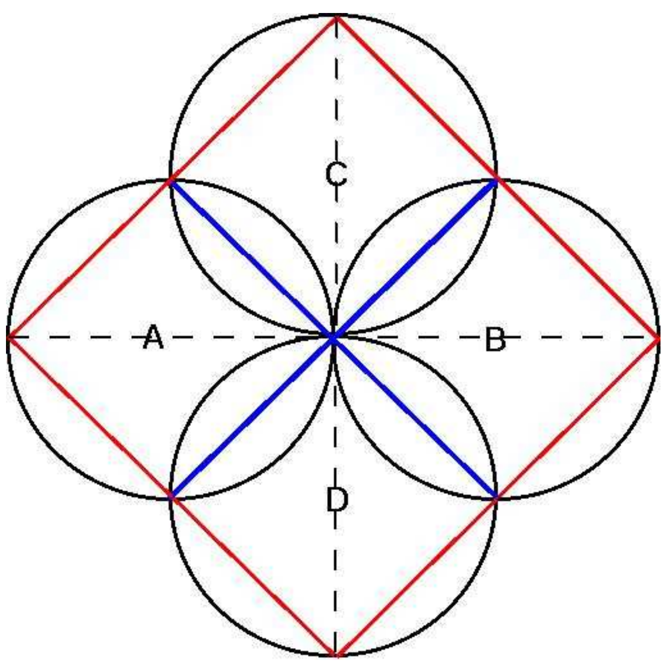


Fig 4 Extra dashed lines, and a red square joining their endpoints, added to the previous figure. The new red square must also pass through the ends of the blue lines, where the circles intersect.

Each pair of blue half-lines of length L meeting at right angles can be taken as two sides of a square. Completing each square with two red lines of length L is another way to produce Fig 4 above, with four axes of symmetry, two diagonal, shown in blue, as previously, and one horizontal and one vertical axis of symmetry shown as dashed black lines in Fig 4.

Each dashed line has two halves, where each half is a diameter of length 2x**R** of one of the four circles. In **Fig 4** the symmetries (reflections) imply that the four blue lines meeting at the center, meet at right angles and that the four quadrilaterals formed by adding two red lines to each pair of blue lines are all squares, as shown in the figure.

Answering the question about areas

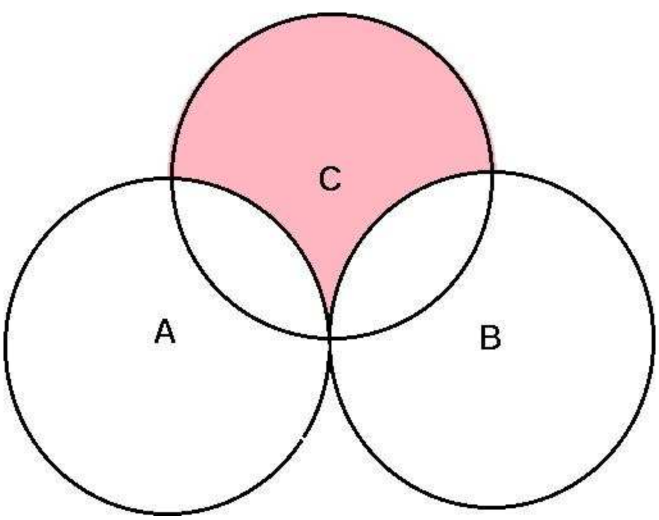
We can now answer our original question about the shaded area in Fig 2 (replicated below on the left of Fig 5).

Figures 3 and 4 show that the shaded area in Fig 2 (=Fig 5) is formed by removing four regions from circle C, where each of the four regions is bounded by a straight line and a circular arc in the new figure on the right of Fig 5.

But the symmetries in **Fig 4** show that the remaining area after removing the four regions from a circle is the same as the area of the square inscribed in the circle, that's because four such segments surround the square in each circle.

Figures 3 and 4 show that the shaded area in Fig 2 (or Fig 5 below) in the circle C is produced by removing from the circle four portions each equivalent to one of the four sectors surrounding the square inscribed in the circle.

So the "strangely shaped" shaded area must be the same as the area of the square that would be obtained by moving two of the shaded circular sectors at the circumference of the circle, into the central region of Circle C, as shown in Fig 5:



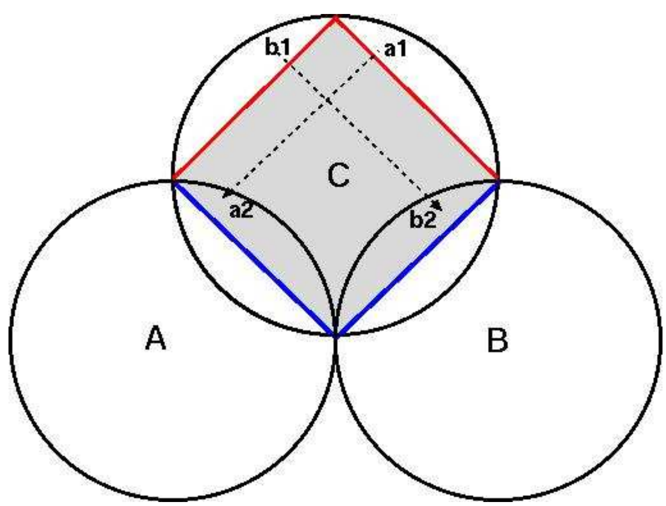


Fig 5 Transforming the shaded region of **Fig 2** (on the left) to a square by moving two sectors of the shaded region at **a1** and **b1** into the unshaded regions at **a2** and **b2**. In this process regions **a2** and **b2** become shaded, and regions **a1** and **b1** become unshaded, leaving a square region shaded. So the shaded region on the left has the same area as the square on the right.

Merely inspecting the original figure (on the left of **Fig 5** does not reveal the equality between regions that is achieved by embedding the figure in the larger construction in **Fig 4**, whose symmetries allow equalities of regions to be guaranteed without using any explicit calculation of areas. Neither does inspection of the left side of Fig 5 reveal that the blue chord common to circles A and C, and blue chord common to circles B and C, shown on the right side of Fig 5 meet at a right angle.

Question

> The derivation of Fig 3 and Fig 4 from Fig 2 can be seen as using a combination of Euclidean Geometry (e.g. drawing circles) and Origami Geometry, using paper folds, for which axioms are given in http://mathworld.wolfram.com/Origami.html.
>
> Is there is a more direct, purely Euclidean, way of showing that the two blue lines in Fig 5 must meet at right angles, without drawing the extra circles and lines?

Now we know that the area requested in the initial problem statement, above is the same as the area of the inscribed square, on the right of Fig 5. We can use Pythagoras' theorem to derive the area of the square, using Fig 6, below.

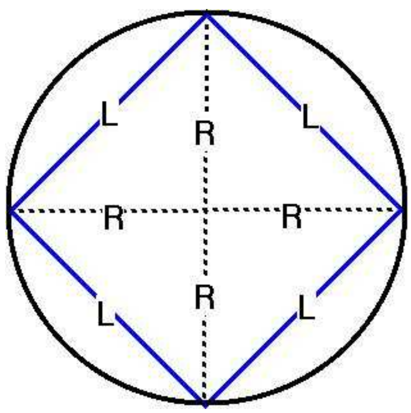


Fig 6 A circle of radius **R** with an inscribed square of side **L**

A square bounded by a circle of radius **R** through the corners of the square is made of four right-angled triangles each of which has two sides of length **R** meeting at right angles, and a third side, the hypotenuse, of length **L**. Given **R** we can compute **L** using Pythagoras' theorem.

$$L^2 = R^2 + R^2 = 2xR^2$$

So the area of the square, and therefore the area of the shaded portion of Fig 2 is simply

$$L^2 = 2xR^2.$$

This result is so simple that the derivation given above is probably unnecessarily complex. Please let me know if you find a simpler derivation.

Two solutions provided by Steve Vickers

https://www.cs.bham.ac.uk/~sjv/

6 Sep 2018

I showed Steve Vickers the problem statement at the top of this file
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/multicirc-problem.html
using this figure, copied from above:

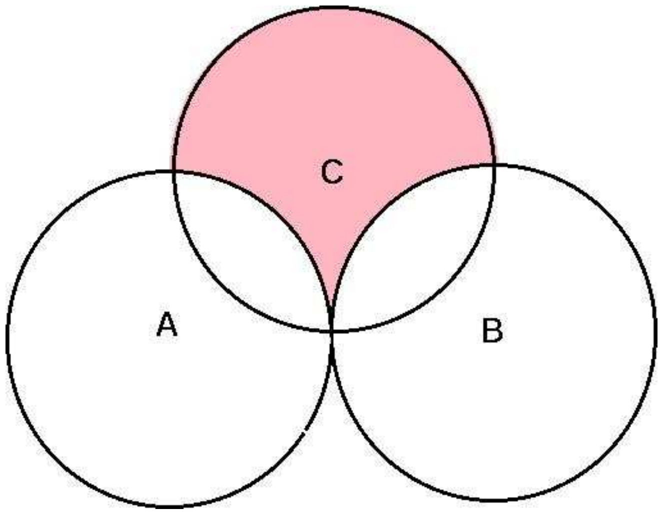


Fig 2a

After which he thought about it then a short time later wrote to me saying (with my inserts in red):

"Here's my solution to your problem of the three circles.

I was trying to think of a slick geometric argument, and failed, so in the train I just went for the calculation.

The area required is the circle C less two lenses.
Half a lens is a quarter circle less half a square, so has area
$\pi R^2/4 - R^2/2$
where R is the common radius of the circles."

Note[AS]:
At first I failed to understand because I thought Steve was referring to half the blue square of side L in
Figure 6, copied here as Figure 6a.

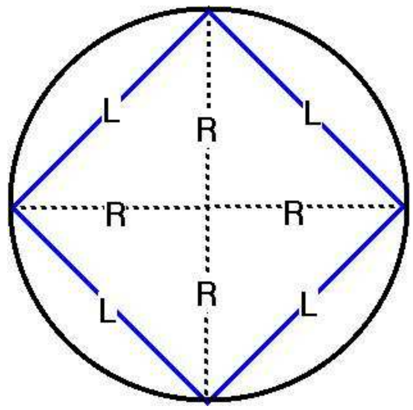


Fig 6a A circle of radius **R** with an inscribed square of side **L**

Then I realised he was talking about the four smaller squares of side R. However, only half of each smaller square is drawn in the figure.

The blue square of side L is made up of four right-angled triangles each of which is half of a smaller square of side R, not drawn.

As Figure 6a shows, the big square of side L has area composed of 4 half squares of side R. So the total area of the big square is twice the area of one small square, i.e. the area of the big blue square is:

$2*R^2$.

Continuing Steve's reasoning:

"Half a lens is a quarter circle less half a square (of side R), i.e. $\pi R^2/4 - R^2/2$

where R is the common radius of the circles.

(It is not hard to see that circle C, in the original figure, is the same size as the other two, and that the intersection lenses subtend angles of 90 degrees at the centre of C.)

Hence the required area is: $\pi R^2 - 4(\pi R^2/4 - R^2/2) = 2R^2$

A problem:

What exactly justifies the claim that the two chords produced by the intersections between Circle C and the two other circles A and B are of the right size to form the side of an inscribed **square**, rather than an inscribed quadrilateral of some **other** shape, e.g. with two short and two long sides meeting?

It is obvious from the symmetry between circles A and B that the inscribed quadrilateral must have a **vertical** axis of symmetry joining the top and bottom corners. But the assumpton that the quadrilateral is a square requires a proof that the horizontal line joining the intersection between A and C and the intersection between B and C is also an axis of symmetry.

In the proof based on Fig 4 the required symmetry came from the construction of circle D adding a new axis of symmetry. Is something equivalent implicit in Fig 5a ?

Steve continued with his second solution:

Having calculated that, I immediately saw a geometric argument. If you slice each lens into two, you can rearrange the four halves around the circumference of circle C and they leave a square hole in the middle. Its area (circle less four half lenses) is the same as the answer we are looking for. The side of the square is $\sqrt{2}*R$ by Pythagoras, so the area is $2*R^2$.

This is essentially the the same as the reasoning given just below Figure 6a, above.

.... It would be interesting to know how other people do it. For myself, I couldn't see the geometric argument until the algebra had given me clues about what to go for. I think I realized straight away that since the πs had gone I could look for a rearrangement with no curved sides, and then the actual answer was easy.

Thanks
Thanks to Manfred Kerber Manfred Kerber for drawing my attention to this problem, after learning of it from Colin Rowat. So thanks also to Colin Rowat.

QUESTION
What sort of brain is capable of discovering and solving problems like this, or understanding the reasoning used above?

What can we learn from this about mathematical consciousness and the mechanisms it uses in humans?

In particular, since I am sure I am not the only person able to work out the proof by thinking about the diagrams presented above, what does that imply about mechanisms of human visual/spatial memory?

What kind of robot brain would enable a robot to discover and solve a problem like this one?

Can any existing neural net model of brain function accommodate such mathematical discovery processes?

This is part of the Meta-Morphogenesis project:
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/meta-morphogenesis.html

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LOOSELY RELATED

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Mathematical phenomena, their evolution and development
(Examples and discussions on this web site.)

http://www.cs.bham.ac.uk/research/projects/cogaff/misc/impossible.html
(or ...../.pdf)
Some (Possibly) New Considerations Regarding Impossible Objects

http://www.cs.bham.ac.uk/research/projects/cogaff/misc/toddler-theorems.html
Meta-Morphogenesis and Toddler Theorems: Case Studies

http://www.cs.bham.ac.uk/research/projects/cogaff/misc/triangle-sum.html
(or ...../.pdf)
The Triangle Sum Theorem
Old and new proofs concerning the sum of interior angles of a triangle.
(More on the hidden depths of triangle qualia.)

http://www.cs.bham.ac.uk/research/projects/cogaff/misc/torus.html
Using Apollonius' construction to find the solution to a "maximum angle" problem.
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/apollonius.html
(or ...../.pdf)
Reasoning About Continuous Deformation of Curves on a torus and other things.

http://www.cs.bham.ac.uk/research/projects/cogaff/movies/ijcai-17/small-pencil-vid.webm
A 17 month toddler discovers and solves a 3D topology problem

http://www.cs.bham.ac.uk/research/projects/cogaff/misc/super-turing-geom.html
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