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**Like most documents on the Meta-Morphogenesis/
Self-informing Universe project**

Multiple Foundations For Mathematics

**Neo-Kantian (epistemic/cognitive) foundations,
Mathematical foundations,
Biological/evolutionary foundations
Cosmological/physical/chemical foundations
Metaphysical/Ontological foundations
Multi-layered foundations
others ???**

**Do we need to understand all of these (and more?)
in order to build artificial mathematical minds
comparable to ancient mathematicians?**

(INCOMPLETE DRAFT: Liable to change)
(Comments and criticisms welcome)

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Abstract: Types of Foundation for Mathematics
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Nowadays, among most mathematicians, the study of Foundations of Mathematics (FOM) is the attempt to identify a portion of mathematics from which the rest of mathematics can be derived. There are also much older investigations of foundations of mathematics that were concerned with explaining how humans could make mathematical discoveries that were non-empirical and were about necessary truths. After explaining these in more detail I'll present various aspects of biological/evolutionary foundations for mathematics that, as far as I know, have never been systematically studied, and will end with a mention of metaphysical foundations, followed by some background notes (noises).

NEW WORK SINCE AUGUST 2017
Added 14 Nov 2017 -- 30 Jun 2019

Since early 2017, I have been looking more closely than previously at some of the kinds of reasoning and biological brain mechanisms required for the ancient discoveries in geometry and topology made by mathematicians like Archimedes, Euclid, Zeno and others, and have tentatively reached the conclusion that to model their discovery and reasoning processes we need a new kind of machine, which I have provisionally labelled a "Super-Turing Membrane Machine". It is not yet clear whether such a machine could be implemented in a digital computer, or whether something comparable to within-synapse brain chemistry is required, as I now suspect. This work clarifies and extends the ideas presented in Sloman (1962,1971,1978a). Detailed requirements specifications for such a machine are still under investigation here:

http://www.cs.bham.ac.uk/research/projects/cogaff/misc/deform-triangle.html
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/super-turing-geom.html
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/bio-math-phil.html
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/turing-intuition.html
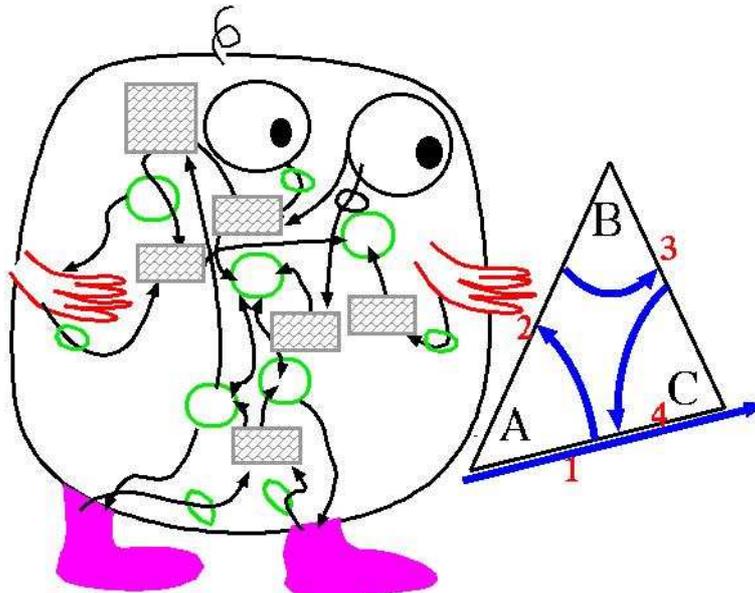http://www.cs.bham.ac.uk/research/projects/cogaff/misc/kant-maths.html

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JUMP TO CONTENTS

How can evolution produce babies that grow up to be mathematicians?

Can we produce baby robots that grow up to be mathematicians? How?
(Displaying proof of triangle sum theorem by Mary Pardoe about 1970)

This document is available in html and (derived) pdf
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/maths-multiple-foundations.html
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/maths-multiple-foundations.pdf

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CONTENTS

Cognitive Foundations

The philosophical question: *"How can mathematical facts can be discovered, conjectured or known?"* is at least as old as Plato (though his answer cannot be taken seriously). Since Kant, this has become a question about **cognitive** foundations for mathematics of interest not only to philosophers and (perhaps) mathematicians, but also psychologists, neuroscientists and biologists.

An answer to this question should explain how it is possible for individuals (a) to understand mathematical concepts and (b) discover, or recognize, mathematical truths expressed using those concepts. Moreover, if, like very young humans, they lack these mathematical abilities, the answer should explain how the abilities develop or are acquired. This requires a theory of cognitive development of mathematical abilities. Additional questions that are left unanswered by this, e.g.

questions about how the mechanisms evolved, are listed below.

A satisfactory theory of mathematical cognition should specify types of mechanism of discovery, learning or reasoning that make it possible for humans (and possibly some other animals and future robots) to make mathematical discoveries. This can be generalised and deepened, because there is not a single transition: complex changes usually need to happen in stages, and in this case intermediate stages may produce or extend precursors or subsets of mature mathematical competences.

An additional requirement is to explain transitions between merely having and being able to use mathematical competences, and being aware of having them: mathematical meta-cognition.

Later transitions, in some cases, produce abilities to help others acquire the competences (using *other-directed* meta-cognition to detect intermediate stages in learning and development).

A further stage may include understanding how the competences work, or perhaps even being able to replicate them in robots: a stage nobody seems to have reached so far.

If mathematical competence is decomposable into sub-competences then it is possible that mature (adult) mathematical competence is not only a product of stages of individual development, but that any biological species whose members have mathematical expertise is likely to have been preceded by evolutionary ancestors with various precursors and subsets of the sub-competences.

The existence of non-human species apparently able to reason topologically and geometrically in planning and executing complex actions to achieve intended results (e.g. nest-building birds) is evidence of this possibility, though exactly which portions of expert human mathematical competence they have is not obvious from their behaviours.

For a very compressed collection of conjectures about evolution of increasingly sophisticated and diverse mathematical mechanisms, based on hypothesised evolution of increasingly powerful construction-kits of various kinds, see Sloman(2017a).

In particular we need to distinguish

(a) having abilities to acquire and use mathematical concepts, e.g.:

"x spatially contains y", "these two lines (or routes) share a common point (or location)", "there is a connected route from x to y", "this line is straighter (longer, thinner, more curved, ...) than that line", "these two collections of items are in one-to-one correspondence", "shape x can be continuously deformed into shape y";

(b) being able to recognise and use mathematical truths expressible using those concepts, e.g.:

"if x contains y and y contains z then x contains z", "if two lines have exactly three distinct points in common then at least one of the lines is not straight", "two circles cannot intersect at exactly three distinct points"[*], "x is continuously deformable into y is a transitive relation", "one-to-one-correspondence is a transitive and symmetric relation".

[*] What about two squashed circles (ellipses)?

(c) Recognizing the **modal** status of these truths (and the falsehoods obtained by negating them): namely the truths and falsehoods are not merely *contingent* truths and falsehoods, they are *necessary* truths and falsehoods as emphasised by Kant (1781), among others.

Analysing these modal concepts, and pointing out that their modality is not correctly analysed in terms of *possible world semantics* but in terms of possible and impossible configurations of portions of *this* world was one of the goals of Sloman(1962).

(d) Understanding that mathematical discoveries about what is possible, impossible, or necessarily the case are not *empirical* discoveries that could be derived from statistical evidence and, if true cannot possibly have any counter-evidence. An example is the discovery that one-to-one correspondence is necessarily a transitive and symmetric relation.

So, widely used statistics-based learning mechanisms may (by chance) suggest mathematical generalisations but they cannot reveal their mathematical character or explain mathematical understanding.

As shown convincingly in Lakatos(1976), this does not rule out the possibility of mathematicians making mistakes, and later discovering and correcting them.

(I have tried to include examples that would not be the first to come to mind for most readers -- partly to indicate the breadth of mathematics, and how mathematical understanding lurks within common sense!)

These comments and examples suggest that a good theory of cognitive foundations for mathematics should be decomposable into a variety of sub-theories

  -- about various sub-competences required
    (including, perhaps, different competences for different areas of mathematics),
  -- about possible developmental trajectories in individuals, and
  -- about possible evolutionary trajectories in which genetic changes occur.

Moreover, it will not suffice merely to identify developmental or evolutionary stages and trails between stages. A good theory should explain exactly how the new biological mechanisms make the mathematical competences possible. This will require mathematical analysis of those mechanisms and competences and their relationships. Merely discovering empirical correlations will not explain how the new mechanisms make new competences possible.

NOTE:
Immanuel Kant frequently asked questions of the form "How is this possible?". I think such questions are at the heart of science as well as philosophy, though their importance is not widely understood (partly because of the unfortunate influence of Popper's demarcation between science and non-science). For more on explanations of possibilities, and a revision of Popper's theory see Chapter 2 of Sloman (1978a), available (here).

Modern vs Kantian foundations
The modern quest for a "foundation" for mathematics by finding a subset of mathematics from which all of the rest of mathematics can be derived is very different from the quest for cognitive

foundations: instead of *mechanisms*, the modern goal is to find a *part* of mathematics from which the rest of mathematics can be derived mathematically.

Any answer will have implications concerning cognitive foundations, though few modern (e.g. in 2017) researchers on mathematical foundations seem to be interested in cognitive foundations for mathematics. (List exceptions...?)

Some of the crucial requirements to be met by an explanatory cognitive mechanism were identified in Kant (1781). In particular, the knowledge acquired must not merely be of learnt *regularities* but must include an understanding of *why* exceptions are *impossible* (i.e. not just highly improbable). Some of the discoveries are of impossibilities:

  E.g.
  6, 8, 9, 10 or 12 blocks can be arranged in a rectangular NxM array with N and M both greater than 1. But that's *impossible* for 7, 11 or 13 blocks. Why?

Kant claimed that such discoveries may be "triggered" or "awakened" by experience but are not *derived* from experience, as empirical discoveries are, which is also why future experiences can refute empirical discoveries but not mathematical discoveries (unless mathematical mistakes have been made -- mathematical cognition is not *infallible* Lakatos(1976)).

Kant claimed, moreover, that the truths discovered mathematically are not merely logical consequences of definitions: in Kant's terminology, they are *synthetic*, not *analytic*. (His definition of "analytic" was insufficiently general and has been rationalised here, following Frege.)

Work on mathematical cognition by psychologists and neuroscientists tends to ignore most of the (philosophical) subtleties and merely investigates which animals can get answers right or at what ages young children do, or which portions of brains seem to be active at the time.

Such researchers on brains and mathematics usually seem to have no idea how the postulated mechanisms support the required mathematical capabilities. (They usually lack the information-engineering skills to design explanatory hypotheses.)

For example, a mathematical understanding of cardinal or ordinal numbers requires a grasp of features of the 1-1 correspondence relationship, including understanding that it is necessarily a *transitive* and *symmetric* relationship. Although logicists were able to use pure logic to produce a definition of 1-1 correspondence and a proof of transitivity, there is no evidence that normal human brains use a logicist concept of 1-1 correspondence. Few of them could follow the required logical proofs. (The original notion of 1-1 correspondence seems to be more a topological concept concerned with structural relationships than a logical notion. (Sloman(1978) Notes to Chapter 8).)

Piaget's research suggests that children do not grasp the transitivity until they are 5 to 6 years old, but I suspect nobody knows how brains have to change to support that understanding. It is not enough merely to learn from statistical evidence that transitivity is usually a property of 1-1 correspondences. As Kant noted, mathematical knowledge does not consist of statistical regularities: it concerns possibilities, impossibilities, and necessities. Neural nets developed so far do not seem to be capable of discovering, or even representing, such modal features of true or false propositions.

Some AI researchers and computational cognitive scientists seek explanatory computational mechanisms at a level of abstraction that is neutral between implementation in brains and implementation in computers, though they test their mechanisms by implementing them on computers, and possibly also seeking mathematical proofs that they have the required properties.

But not everyone assumes that cognitive foundations must be computational foundations, in some cases because of their beliefs (often incorrect) about what computers cannot do. (E.g. John Searle and some of his admirers mistakenly believe that computation is purely syntactic and cannot involve semantics: ignoring the semantic features of programming languages that are essential for their causal roles in computers.)

My own view is that it is difficult to characterise the computational requirements for artificial mathematicians to meet Kant's conditions, and that it is an open question whether current computers have the right features, or whether very different forms of computation are required. This applies especially to the mathematical discoveries in geometry and topology made several thousand years ago, many of them reported in Euclid's *Elements.*

I have begun to specify an alternative to Turing machines or digital computers i this paper on a Super-Turing Membrane machine, motivated by examples of discoveries in Euclidean geometry and topology.
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/super-turing-geom.html

I'll now present an informal high level overview of varieties of foundation for mathematics, including several kinds of foundation not yet mentioned. In the recent past (a deliberately vague reference) only the mathematical foundations are normally discussed by philosophers or mathematicians who claim to be interested in foundations of mathematics. But I'll start with older types.

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(1) Neo-Kantian (cognitive/epistemic) foundations
This is a modified version of Immanuel Kant's attempt to describe basic features of mathematical minds. They are able to have certain sorts of experience on the basis of which they can discover and prove various kinds of mathematical truth, including truths of arithmetic, topology and geometry.

Kant offered what we might label "cognitive foundations" for mathematics by describing features of minds that enable them to understand mathematical concepts and discover and make use of mathematical theorems and proofs. He claimed that the knowledge obtained in this manner was *non-empirical*, included *necessary truths*, and was *synthetic* -- i.e. not derivable from definitions using only logical inferences.

This three-fold characterisation combines a theory of the nature of mathematical truths with a theory of the features of (natural and artificial) minds that enable them to discover mathematical truths. My 1962 DPhil Thesis was an attempt to defend Kant against criticisms that were common in Mid 20th Century, e.g. after physical space had been found to be non-Euclidean. Far from being a refutation of Kant, that discovery merely enriches what he needed to explain, namely the existence of (at least) three very different ways of extending a core subset of Euclidean geometry, to produce elliptical, hyperbolic and Euclidean geometries, all worth investigating mathematically.

A less dramatic, little known, modification of Euclidean geometry, the "neusis" construction Sloman(2015), makes it easy to trisect an arbitrary angle, although that was proved to be impossible using the constructs in Euclidean geometry. I am sure Kant would have regarded that discovery as fitting his theory of mathematical cognition, if he had known about it.

I don't know whether Kant thought it possible that some kind of non-human mind could in principle exist with the mechanisms required for making mathematical discoveries. A good theory of the sort he was trying to construct should be applicable to a variety of types of mind, including future highly sophisticated artificial minds.

One of the claims that could be read into Kant's ideas is that the discovery mechanisms in human mathematicians are infallible. However, as far as I know he did not make that claim, and the work of Lakatos (1976) shows that far from being infallible even the greatest mathematicians can make mistakes that can later be discovered and, in some(!) cases, corrected.

Kant lived before it was possible to specify minds in computational terms, though it seems to me that he was describing requirements for such a specification and moving towards such a specification, though his version of computation, as far as I know, was not restricted to *discrete* computation like modern computational foundations. For example, his claim that we can discover that there are non-superimposable 3-D structures, such as a right-handed and a left-handed helix does not specify a form of computation, but this intuitive discovery does not seem to rest on discrete operations on discrete symbols.

However he was not very clear about the alternatives available, and neither has anyone else been, as far as I know.

Examples of Kantian reasoning
I have been trying to assemble candidate examples of experiences on the basis of which humans can discover and prove various kinds of mathematical truth, including truths of arithmetic, topology and geometry.

Added: October 2017
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/deform-triangle.html
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/triangle-sum.html
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/torus.html
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/shirt.html
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/trisect.html
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/triangles.html
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/toddler-theorems.html#pencil

One important modification is a Lakatos-inspired alteration from (a) trying to explain how it is possible to know that space (or experienced space) is Euclidean to (b) trying to explain how it is possible to know that there are at least three distinct ways of experiencing space: Euclidean, elliptical and hyperbolic (all easily illustrated in 2-D surfaces) with a common core and one axiom different, and to derive many theorems common to all, and some true only in a subset.

(I suspect that if Kant had known about non-Euclidean geometries he would have modified some, but not all, of his examples.)

Another requirement is to explain how a mathematical mind (like Archimedes' mind?) can discover a simple extension to Euclidean geometry, the neusis construction, that makes it easy to trisect an arbitrary triangle.
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/trisect.html

A feature of (a) will be accommodating the evidence from Lakatos that although the mathematical discoveries are non-empirical, mistakes of various kinds are possible, and may need to be repaired.

(1a)Proto-Cognitive foundations In the context of evolutionary theory, it is to be expected that subsets of the cognitive mechanisms making human mathematics possible will have occurred in evolutionary ancestors of humans, and in some of their non-human descendants, i.e. intelligent non-human animals, such as squirrels, elephants and crows. Some of the competences may have evolved independently in different evolutionary lineages.

Many organisms and biological processes (some mentioned below) make use of information about mathematical structures but lack the abilities of human mathematicians to reflect on and reason about the information being used. Biological evolution must have produced such abilities long before the full Kantian (e.g. human-like) mathematical abilities. This category can be divided into many different sub-cases (a task begun below).

Pre-verbal human toddlers seem to be capable of discovering and using some kinds of mathematical information (e.g. topological information) long before they are capable of being aware of what they are doing or able to communicate it. This may be similar to mathematical competences of non-human animals, e.g. nest-building birds, or a toddler apparently reasoning about possible 3D trajectories for a pencil in this video.

NOTE: Many researchers in biology, psychology or neuroscience, assume that all mathematical cognition is concerned with numbers. However, there are many non-numerical aspects of mathematics, including topology, geometry, logic, grammars, theory of computation, group theory, for example. It is not widely recognized that many organisms, including pre-verbal children, make use of some of these non-numerical mathematical competences, especially topological and geometrical competences (used in perception of positive and negative affordances as discussed below). I think Piaget understood some of this. Compare Sauvy(1974), inspired in part by Piaget's work.

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(2) Mathematical foundations for mathematics
Is there a subset of mathematics from which all of the rest of mathematics can be derived (mathematically)? For many mathematicians and philosophers of mathematics the word "Foundations" now refers to an assumed part of mathematics from which all the rest of mathematics (or as much as possible) can be derived rigorously. It is not obvious that there is any finite, mathematically specifed, subset that meets this condition.

During (approximately) the last two centuries, many mathematicians, logicians and philosophers have attempted to find a (preferably finite?) subset of mathematics from which everything else can be derived mathematically. Examples include thinkers such as Boole?,Peano, Frege, Dedekind, Cantor, Russell and Whitehead, Hilbert, Brouwer (out on a limb for a while, but now increasingly influential?) and many others including several whose work I have not encountered!

In this context, logic, often thought of as part of philosophy, and taught as such, is now usually regarded as a part of mathematics (including use of logical inference techniques).

Since the early days of this enterprise many different sorts of mathematical foundation for mathematics have been proposed, including various suggestions for logicist foundations, intuitionist foundations, set-theoretical foundations, computational foundations, and others, on which I lack enough knowledge to comment.

There are different candidates for mathematical foundations and different opinions about them among mathematicians, including disagreements about whether a unique mathematical foundation for all of mathematics is possible.

A multiplicity of minimal foundations seems most likely to me. Compare the (mathematical) equivalence of Turing machines, Lambda calculus, Recursive function theory, and Post's production systems, for foundations of the theory of computation?

It is not obvious whether there can be a minimal, or in some sense optimal, non-finite foundation -- namely a part of mathematics from which everything else can be derived, but which loses that generative power if anything is removed from it.

In principle there could be a unique minimal foundation or there might be several alternative minimal foundations, each of which would have to be capable of proving all the others.

An infinite foundation is trivially possible -- just combine all mathematical results into one system. (This might involve a mish-mash of formalisms.)

It seems that there cannot be a **finite** foundation that enables any mathematical truth to be proved in a finite number of steps. If there were, all mathematical truths could be enumerated, which Cantor showed to be impossible.

For further details and examples, see [Sakharov (2003)](). I shall not discuss mathematical foundations for mathematics in detail.

(2a) Meta-foundational foundations for mathematics
A meta-foundational mathematical theory could provide a (mathematical? philosophical?) framework for analysing and comparing different proposed mathematical foundations for mathematics. Discussions of this type are often part of discussions about rival proposals for mathematical foundations.

(3) Biological/evolutionary foundation(s)
What made it possible for mathematical discoveries to be made and used by products of biological evolution, and later reasoned about and discussed -- using up to four or five layers of meta-cognition (to be explained here later).

Over billions of years, products of biological evolution have implicitly used mathematical discoveries.

For example, many homeostatic control mechanisms (using negative feedback for control) depend on mathematical relationships between current state, selected change, and target state. These "discoveries" were initially made and used blindly: nothing existed at the time that could be

described as understanding the problems and the solutions and how the solutions worked.

Some later products of evolution (i.e. some humans) developed abilities to notice, reflect on and explain mathematical discoveries, including human scientists and engineers who re-discovered some of the principles previously discovered and used by evolution. These humans were not aware that biological evolution had made and used some of the discoveries much earlier.

Negative feedback control systems are just one type of mathematical discovery implicitly made by biological evolution. Others involved discovery of mechanisms for encoding information acquired by individuals in such a way that they could build up re-usable stores of knowledge, in some cases also using the same information structures to formulate percepts, goals, questions, generalisation, predictions, plans, etc. This required evolution of internal languages with structural variability and compositional semantics, long before similarly powerful languages were used for communication Sloman 2008/2011.

Various mathematical sub-stages can be distinguished, including the following, which must have occurred before evolution produced the abilities exhibited by ancient mathematicians such as Euclid:

(3a) evolution of chemical/physical mechanisms whose mathematical properties served biological functions -- e.g. reproductive encodings, discussed in Schrödinger(1944);

(3b) evolution of means of encoding *information* about such mechanisms that could be used to replicate them in biological reproduction;

(3c) evolution of various parameterised mathematical abstractions that allowed general characterisations to be instantiated in different ways, e.g. supporting changes during growth of an individual, or supporting variations across different species with different needs, different morphologies and different behaviours;

(3d) evolution of cognitive abilities allowing individuals to take mathematically based decisions when processing information (and taking them much more quickly than evolution could); and later still

(3e) evolution of meta-cognitive competences allowing organisms to become aware of what they were doing in taking those decisions, so that they could evaluate their mathematical reasoning abilities, communicate them to others, challenge them, defend them, etc.

Development in individual humans seems to involve a related but different set of sub-stages.

There seem to be two very different kinds of biological foundation.

1. Insofar as biological evolution "discovered" various mathematical abstractions that had important practical applications, and explicitly separated them from other parts of genomes specifying designs for organisms, it can be said to have made and used mathematical discoveries, even though there was no conscious agent intentionally doing any of that.

   An example would be the discovery that negative feedback control loops can be used to maintain important biological states (temperature, osmotic pressure, direction of motion, etc). If some genomic pattern corresponding to that sort of mechanism can be found that is repeated

across species, but using different parameters and different contexts, or even if the distinction can be found in a genome that specifies a control mechanism that uses variable inputs and outputs where the outputs are mathematically related to the inputs, e.g. via negative feedback, then we can say that evolution implicitly and unwittingly made and used a mathematical discovery, indeed one that was later rediscovered and used by human engineers in many engineering contexts, including the Watt governer.

2. Alternatively or in addition evolution could have produced (and did produce!) individuals capable of discovering that machines they were building could be given homeostatic control mechanisms that allowed temperature, pressure, speed, direction of movement or other physical properties to be controlled. Insofar as they understood how the negative feedback worked, those individuals were making and using mathematical discoveries, which they could pass on to others if they wished.

Case (2) could be regarded as evolution making a meta-mathematical discovery.

Some of the details of developmental/epigenetic mechanisms required for these competences are presented in this discussion paper
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/meta-configured-genome.html (also pdf)
The Meta-Configured Genome: Multi-layered, multi-stage, epigenesis (with Jackie Chappell).

NOTE: David Mumford's online archive
This includes a discussion of some of the problems of getting biologists (I would include neuroscientists and psychologists) to understand mathematics relevant to their work:
http://www.dam.brown.edu/people/mumford/blog/2014/Grothendieck.html
E.g. see the Comments section.

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(4) Physical/chemical foundations

[I don't know whether this is being investigated by anyone else, though Turing seemed to start on something that could have led in this direction shortly before he died.]

How does the physical/chemical universe make possible, and constrain, the kinds of mathematics required for its description and how does it make possible the production, by evolution or engineering, of types of machines (including organisms) with abilities to discover, make use of, and in some cases reason about the mathematical features. Some partial answers are given in
Schrödinger(1944)

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(5) Metaphysical/Ontological foundations?
This kind of study attempts to answer the question: what makes it the case that there are mathematical truths, some or all of which can be discovered and used, whether by human mathematicians or anything else, e.g. biological evolution or its products.

In principle, this could be further split into different sorts of foundation or grounding. For example, there might be a world whose physical/chemical properties could not support evolution of organisms with brains capable of making and organising mathematical discoveries of certain kinds. Could there, for example, be universes in which brains could evolve that are capable of making discoveries in geometry and topology but not the arithmetical discoveries that depend on the use of mathematical induction, or proof by contradiction?

Are there aspects of the kind of mathematics discoverable (and possibly usable) in **our** universe that would be applicable to **all possible** physical universes, and some aspects of mathematics that are restricted to a **subset** of possible universes with special properties?

For example, could there be a kind of universe that does not support the physical mechanisms (e.g. brain mechanisms) required for discovery or invention of Euclidean geometry or its alternatives?

There may be even more limited physical universes in which it is not possible for physical information processing mechanisms to exist that can support the discovery of the full set of natural numbers, even if some subsets are found to be useful. In that sort of universe no brain mechanisms could ever construct even the thought that the natural numbers "go on indefinitely".

It is not obvious what sort of brain could grasp the usefulness of counting using a fixed list of counting noises in connection with a wide range of tasks, and be incapable of having the thought: there is no largest collection of objects that can be counted. Perhaps some of our ancestors were at such a stage.

In our universe not all brains seem to have that capability, and it is not clear at what stage of development human children are able to comprehend such thoughts, nor how their brains need to change during development to give them such abilities. A partial analysis of computational mechanisms required during development of arithmetical competences was offered in Chapter 8 of Sloman (1978a), available online here.

CONTENTS

Personal note: 1962 DPhil thesis defending Kant on mathematics

My 1962 DPhil thesis attempted to present a defence and elaboration of what I took to be the (partial) answer presented in Kant (1781). Using examples from geometry, arithmetic and informal topology, he defended mathematical knowledge as

- non-empirical, i.e. not capable of being refuted by things or facts discovered empirically (e.g. contra J.S. Mill)
- synthetic (non-analytic), i.e. not derivable by logic from definitions -- (contra David Hume [perhaps])
- concerned with necessary (i.e. non-contingent) truths, unlike common sense generalisations, history, and (most) scientific knowledge.

  (In 1962, I was academically very naive, and did not realise that after being deposited in a library, a thesis should be published in book form in order to be noticed. A digitised transcript is now freely available online here.)

Kant's work also (implicitly) contradicted a view of mathematics as a man-made collection of socially accepted "results", expressed by Wittgenstein as
    "For mathematics is after all an anthropological phenomenon" [Wittgenstein(RFM)],
or in more recent forms discussed in [Gouvea(review)]. Such claims seem to imply that humans could have chosen to create mathematics with different contents, contradicting current

mathematical theorems, etc.

It seemed to me that Kant had shown, at least in outline, that many familiar mathematical discoveries were concerned with properties and relationships of various spatial and abstract structures. These discoveries could be made by minds with an appropriate architecture, which he attempted to characterise. He could not have understood in 1781 that he was trying to do something that required the conceptual and engineering resources of a very advanced form of Artificial Intelligence. (AI is still not sufficiently advanced in 2017.)

Examples include the discovery that three planar surfaces cannot enclose a space, that a right hand glove cannot fit on a left hand unless turned inside out, that the different coexisting features of a large object (e.g. a house) will be experienced in a particular order by a moving perceiver, where the order depends on the trajectory of motion. E.g. reversing the direction of motion around a building will reverse the order in which windows and other features are experienced. Had he studied the mathematics of relations he might have used as an example the discovery that one-to-one correspondence (bijection) is necessarily a transitive relation: a feature that is essential for practical applications of the natural numbers (cardinals and ordinals).

> **Note:** In the thesis I rejected the "possible worlds" interpretation of the modal concepts (e.g. "necessarily", "cannot", "possible"), and instead suggested that they refer to relationships of compatibility, incompatibility, entailment, etc. between features that can occur in *this* world. Discovering that a sequence of logical formulae can, or cannot, be generated by a particular set of logical rules would be a special case of such "geometrical" (but not Euclidean) mathematical discoveries.

Do humans create mathematics or discover mathematical truths?

A claim that seems to be widely accepted (or at least widely discussed) has always seemed to me to be deeply mistaken, namely the claim that mathematics is a creation of human minds, mentioned above.

However, rejecting anthropologism is compatible with accepting Kant's view that only certain sorts of minds, including (normal) human minds and possibly other non-human kinds of minds, can make mathematical discoveries in geometry, topology, arithmetic, etc.

That suggests a need for a theory explaining exactly what sorts of mind can make various sorts of mathematical discovery. Kant tried to produce a generic answer for all minds but (in my opinion, and his) lacked a sufficiently powerful theory of mental mechanism. I suspect he would have welcomed the opportunity to use AI concepts, tools and techniques for that project, though it is not obvious that the current tools and forms of computation suffice, for reasons suggested in Sloman (1971) and more recent papers.

Kant's notion of a "schema" seems to have anticipated the notion of a certain type of computer program, including programs that parse sentences and produce descriptions of their grammatical structures, or which analyse images and produce hierarchical descriptions of their parts and relationships.

In both cases something more is required to produce semantic interpretations: what is referred to, described, or depicted. (However, I am not a Kant scholar and have no wish to get involved in exegetical debates.)

Extending James Gibson on "affordances"

About two hundred years after Kant, James Gibson (1979) took an important step, by rejecting the notion that biological perception is merely concerned with taking in information about what is the case in the immediate environment, e.g. where the visible or touchable surfaces are located in 3-D space, which surfaces are parts of the same object, and what their 3-D relationships and relative motions are.

He proposed that perception mainly provides organisms with information about affordances for the perceiver -- that is, information about possible actions that perceivers are able to perform in the current environment, and what the consequences of those actions (or doing nothing) would be.

Gibson's analysis can be expanded by acknowledging that perception provides information about a much wider variety of *possibilities* for change in the environment and *constraints* on those possibilities. I have called the possibilities for change and constraints on change detected independently of any practical relevance to the perceiver "proto-affordances" e.g. in Sloman(2008). (I suspect the importance of this extension of Gibson's ideas has been recognized independently several times.)

In other words, perception provides information about possibilities and impossibilities without being restricted to information about what the perceiver can and cannot do, and what the resulting benefits or costs for the perceiver would be Sloman(1983), Sloman (1986), Sloman (1996).

It also provides more general information about what can and cannot happen, including information about which visible objects or parts of objects can move in relation to one another and what the consequences can and cannot be. This may include information about what other agents can and cannot do ("vicarious affordances").

This ability to acquire information about what changes can and cannot occur in perceived spatial configurations, may have been one of the evolutionary precursors to human abilities to make mathematical discoveries in geometry and topology. For example, Liebenberg(1900), argues that *"a human scientific intellect that became capable of dealing with the subtleties of mathematics and physics evolved at a time when humans were still hunter-gatherers"*, because of the perceptual and reasoning powers required for successful tracking.

We should not be surprised to find closely related capabilities in other intelligent animals, even if they lack the meta-cognitive abilities to notice, think about, and communicate what they have learnt, which humans can do.

Can we make artificial mathematicians?

After I encountered artificial intelligence (through Max Clowes, an inspiring researcher on machine vision) around 1969 I hoped it might be possible to provide the required cognitive foundation by showing how to build a "baby robot" able to develop mathematical competences of the sorts Kant had described -- with information processing mechanisms that allowed experience to "awaken" or "trigger" mathematical discoveries that are not *derived from* or *based on* experience. (Loosely

paraphrasing Kant).

This has proved extremely difficult and has not yet been achieved. As far as I know, no AI system is even close to being able to make the discoveries made by ancient mathematicians.

I don't know whether that's because virtual machines running on digital computers cannot be given the powers required to replicate virtual machines that run on biological brains, or simply because we have not yet worked out how to use digital computers as a basis for engines (virtual machines) that can do geometrical and topological reasoning (e.g. discovering the geometric and arithmetic truths and proofs in Euclid's *Elements*, or topological impossibilities like these http://www.cs.bham.ac.uk/research/projects/cogaff/misc/rubber-bands.html).

A few years ago, partly inspired by Turing's work on morphogenesis published two years before he died, Turing(1952), now his most highly cited paper, and a remark about the importance of chemistry in brains, in Turing(1950), I decided to investigate whether we might be missing something produced by biological evolution. This requires attempting to identify previously unnoticed evolutionary transitions in *types of information processing* (e.g. transitions required by changes in environments, changes in bodily structure, changes in opportunities and dangers, etc.) that might give clues regarding previously unnoticed mechanisms selected by evolution. Perhaps some undiscovered mechanisms that evolved long ago are still playing important roles in animal brains.

So the "Meta-morphogenesis" project, first described in [Sloman M-M], was informally begun in 2011, searching for previously unnoticed changes in types of information processing produced by evolution, that might have met new mathematical requirements, using mechanisms that could still be operating unrecognised in animal brains. (The prefix "meta" referred to the fact that the products of biological evolution constantly change the mechanisms of biological evolution.)

The project could alternatively be named "The self-informing universe" project, because a major feature of evolution is constantly extending (a) the varieties of types of information processing, (b) the varieties of types of information content that products of evolution can acquire and use, and (c) the variety of information processing mechanisms serving these goals.

After a while it became clear that natural selection lacked the power to explain the kinds of novelty required, although it explained how novel options could be adopted if available. The gap could be filled by biological construction kits, however.

This led to a (still growing) theory of construction kits, including the fundamental construction kit (FCK) provided by the physical universe and many derived (evolved) construction kits (DCKs) of various kinds, including both concrete and abstract construction kits. See Sloman(2017a). (Evolved types of scaffolding were also required).

New construction kits made various evolutionary transitions possible by re-using previously evolved mechanisms in new contexts, without which the full multi-step evolutionary process from the earliest type of organism would have to be replicated for each new species using an instance of the previously evolved design. Re-usable construction kits and parametrised abstractions make possible evolutionary histories with multiple branches.

This need for the theory of natural selection to have explanatory power supported by appropriate mechanisms was also recognized by M.W. Kirschner and J.C. Gerhart (2005), and more recently by the "Constructor Theory" of David Deutsch and colleagues (although I have not yet checked out details).

The ability of evolution to create increasingly powerful re-usable construction kits implied the implicit ability to detect their possibility and their possible applications.

This required several types of (implicit) mathematical discovery: discovery of new parametrised abstractions that could be instantiated to produce old and new instances, discovery of new possibilities for actions, events and processes, including new possibilities arising out of new uses of old possibilities. In some cases, new useful construction kits were discovered/invented.

All of these discoveries altered the mathematical powers of evolution and its products, and many of them would have provided new biological information processing capabilities, including new uses of mathematical abstractions in perception, action control, learning, planning, collaborating, reproduction ....

Example: In 1944 Schrödinger pointed out the important mathematical features of quantum mechanics that enabled long multi-stable molecules to have properties required for genetically encoded information transferable across several generations) allowing for *huge* amounts of discretely controlled variability. (Anticipating Shannon?) Schrödinger(1944)

These features could justify a view of Evolution as a "Blind mathematician" rather than merely a "Blind watchmaker". (Paley?) Many of the discoveries were of mathematical possibilities (i.e. some collection of existing mechanisms made possible something new and useful). Implicit proofs of those possibilities were (blindly) provided in the evolutionary trails leading up to the new constructions. I suspect there are many important trails that we have not yet discovered, especially trails leading to brains of mammals, birds, reptiles, and the amazing Portia spider.
https://en.wikipedia.org/wiki/Portia_%28spider%29
(Also Tarsitano and Jackson (1997) and Tarsitano(2006)).

CONTENTS

WARNING: Work in progress
The remainder of this document is liable to be altered drastically with some parts removed and others added.

A panel discussion in Edinburgh, in Sept 2016 http://conferences.inf.ed.ac.uk/blc/, with participants Michael Atiyah, Julian Bradfield, Alan Bundy, Mateja Jamnik, Ekaterina Komendantskaya, Maurizio Lenzerini, Phil Scott, and myself, and related email discussions with Michael Fourman, triggered another attempt at clarification (after more reading).

After the conference, I sampled online views of mathematicians (including Atiyah, Manders and others) about the nature of mathematics, and about several different sorts of foundation for mathematics.

- Cognitive (e.g. Kantian) foundations [requirements for individual mathematical thinkers, of various sorts.]
- Mathematical foundations (some hoped for core subset of mathematics from which the rest can be derived -- the dominant form of current research on foundations of mathematics?)
- Biological evolutionary foundations: various processes and intermediate products that made it possible for natural selection to produce increasingly sophisticated biological uses of mathematics including (eventually) mechanisms that meet the requirements for cognitive foundations (e.g. what made it possible for evolution to produce a species including Euclid, etc.) (Could there be a universe in which a Euclidean mind can evolve but not the mind of Cantor? Or vice versa?)
- Cosmological/physical/chemical foundations: the features of the universe that made possible the biological processes leading via many stages, with the help of increasingly complex evolved construction kits, to mathematical minds of various kinds and degrees of sophistication.

   David Deutsch tries to answer some of these questions in his 2011 book, But I think he ignores the special requirements for minds like Euclid, Archimedes, and other ancient geometers.

- Ontological/metaphysical foundations:
   Relationships between possible physical universes and

   (a) the kinds of mathematics required to describe them and

   (b) the kinds of mathematical information processing abilities they can produce.

   E.g. could there be a universe in which the most sophisticated life forms possible could not have produced Euclid's elements?

   [Conjecture: if so, that universe also could not have produced single celled organisms found on earth.]

Cognitive/Epistemic vs Mathematical Foundations

For many years I have been interested in Cognitive Foundations for mathematics, rather than Mathematical Foundations.

The ideas grew out of work on my Oxford DPhil thesis Sloman (1962), defending Kant's philosophy of mathematics Kant (1781), in particular his claim that mathematical discoveries provide knowledge of propositions that
(a) are necessarily true or necessarily false (non-contingent),
(b) have truth values that cannot be derived from definitions using logical reasoning (i.e. they are, in Kant's terminology, non-analytic = synthetic),
(c) their truth cannot be derived from or refuted by empirical observations (i.e. the knowledge is non-empirical.

   These distinctions were summarised in Sloman(1965), and discussed in more detail in my thesis Sloman(1962). This required extending Frege's notion of function by allowing values to be determined by a combination of the arguments given and the state of some part of the environment. These context-dependent functions were called "rogators", as explained in

Sloman(Rogators).

These ideas were used to defend Kant against philosophers I met in Oxford who claimed that his ideas about mathematics had been proved wrong by Eddington's empirical evidence (in 1919) in favour of Einstein's General theory of relativity, showing empirically that physical space is non-Euclidean. I argued that removing the parallel axiom and its consequences left a significant body of mathematical knowledge supporting Kant. An early presentation by Lakatos on the history of Euler's theorem (later published in Lakatos (1976)) had already pointed out the importance of distinguishing Kant's claims from a claim that mathematicians are infallible.

Nevertheless, I felt that my defence of Kant left too much unexplained. So after learning about Artificial Intelligence (thanks to Max Clowes) around 1969, especially work in machine vision, (see Clowes Tribute), I began to think about how Kant's ideas could be supported by working AI models (e.g. a "baby robot" that "grows up" to become a mathematician using mechanisms and developmental stages consistent with what I took to be Kant's views). I hoped that in that way Kantian ideas about the foundations of mathematics would grow into computational theories.

This contradicted the claim in [McCarthy & Hayes 1969] that logic-based forms of representation would meet the metaphysical, epistemological, and heuristic requirements for intelligent robots. (They did not explicitly include robot mathematicians with the intelligence of Euclid, Archimedes, etc.) In Sloman (1971) and various subsequent papers I tried to suggest that at least the heuristic requirements were in some cases better served by non-Fregean, "analogical" forms of representation, which, contrary to popular beliefs, need not be isomorphic with their referents, e.g. 2D pictures of 3D scenes used for planning and controlling actions -- using information processing mechanisms in brain that are still not understood, and cannot be discovered by current methods of brain research.

An example of what needs to be explained is a 17.5 month pre-verbal toddler apparently formulating and testing a conjecture in 3-D topology with no social interaction [here]. (I just happened to notice that something interesting was happening and had a low-quality video camera available.)

Priority of internal (generalised) languages

There are aspects of intelligence in pre-verbal humans and many non-human animals (e.g. squirrels, crows, elephants, monkeys) that seem to be precursors of mathematical intelligence in humans, suggesting that information-processing machinery used by mathematicians (including ancient mathematicians) might have evolved from earlier forms of mathematical machinery used in many other species.

This implies that long before structured languages were used for communication they must have been used for internal purposes, including processing sensory (or more precisely, sensory and perceptual) information, forming intentions, questions, hypotheses, plans, re-usable memories, and controlling complex actions, as argued in Sloman(Inner[1978]) and, in much more detail, in Sloman(Glang[2011]).

This claim is very different from Fodor's theory of "A language of thought"(LOT), Fodor(1975) which claims that each individual is born with a genetically provided language that is rich enough to express all semantic contents required throughout life. I.e. all other languages used by an individual must be translatable into the LOT. There is not a shred of evidence to support

that theory or that evolution would be capable of producing such genetic information -- e.g. a specification for a language capable of expressing all current and future scientific theories and mathematical conjectures or theorems.

In my experience, very few researchers are able to take seriously the idea of (non-Fodorian) internal languages being required by pre-verbal human ancestors and other intelligent animals. Most somehow seem to have been brainwashed (by our culture?) into assuming that languages "by definition" are used for communication and that information "by definition" is something transmitted -- ignoring the fact that *use* of information must be more fundamental than transmission, since transmitting something useless would be pointless.

One of the few exceptions I know of is the mathematician David Mumford, who seems to have arrived at the same idea (inspired by Philip Lieberman's work) Mumford (2016). I would express it differently. Instead of saying "grammar isn't part of language", we should say "internal languages needed grammatical structures (not necessarily linear or 2D structures) and also compositional semantics, long before external languages with those features existed". This requires an extension of the concept of "language" which is already widely used to refer beyond spoken and written human verbal languages, e.g. the language of flow-charts, programming languages, the language of circuit diagrams, the languages of various sorts of maps. All of these usages extend the notions of syntax and semantics beyond linear strings of discrete items.

Limitations of current AI

Unfortunately, the AI programming techniques developed so far do not seem to be able to capture all the required aspects of mathematical and proto-mathematical intelligence in humans and other animals. But the project has not stalled: I have been collecting and analysing many examples of mathematical discovery (including very simple discoveries, and discoveries apparently made by pre-verbal children and other animals), with the hope that detailed study of some of the examples will give clues regarding required computational mechanisms and formalisms (probably including new sorts of virtual machinery). [See below]

Thanks

Some of the work since early 2015 was triggered by (mostly email) discussions with Aviv Keren [AK], though he cannot be blamed for any of my errors or confusions. I have had related discussions with several colleagues and students, over many years.
[AK] https://www.researchgate.net/aviv.keren

Additional stimulation came from a panel discussion in Edinburgh, in Sept 2016 http://conferences.inf.ed.ac.uk/blc/, with participants Michael Atiyah, Julian Bradfield, Alan Bundy, Mateja Jamnik, Ekaterina Komendantskaya, Maurizio Lenzerini, Phil Scott, and myself, and related email discussions with Michael Fourman. After the panel discussion I sampled online views of mathematicians (including Atiyah) about the nature of mathematics, and several different sorts of foundation for mathematics.

Thinking about where I agreed and disagreed with various opinions, provided what felt like a deeper, but still incomplete, understanding of how questions about the nature of mathematics and the possible answers could vary. I now realise that there are several more candidates for foundations, or partial foundations, for mathematics than most researchers have noticed. The list presented here is still tentative.

A partial list of discussion notes on evolution, mathematical cognition, vision, learning, development, mathematical development, types of information processing, philosophy of mind, and related topics is here:
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/AREADME.html

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CONTENTS

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