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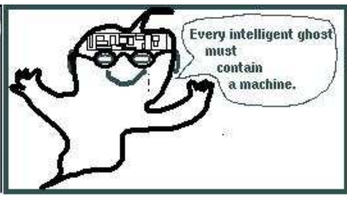

Every intelligent ghost must contain a machine

An information-processing machine

(DRAFT: Liable to change)

Key Aspects of Immanuel Kant's Philosophy of Mathematics

Ignored by most psychologists, neuroscientists and AI researchers studying mathematical competences

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This paper is
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/kant-maths.html
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/kant-maths.pdf

A new, shorter, document with overlapping contents is
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/kant-hume-mathematics.html

This is a companion-piece to a discussion of Turing's notion of mathematical intuition:
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/turing-intuition.html (also pdf)

I am grateful to Timothy Chow (http://timothychow.net/) for probing questions and criticisms of an early draft of that paper that made me appreciate the need for a separate outline of Kant's philosophy of mathematics, at least as I interpret it in the context of asking whether Turing had reached fundamentally similar conclusions about the nature of mathematical discovery in his distinction between *mathematical intuition* and *mathematical ingenuity*, discussed in the above paper -- also work in progress.

I wrote a thesis (never published, but now freely available online) defending Kant's philosophy of mathematics a long time ago Sloman(1962), but I am not a Kant scholar. For views of established Kant Scholars, see, for example, Posy(Ed, 1992). What's most important for my purposes is not whether the claims made here about spatial mathematical knowledge were previously made by Kant, but whether they are *true*. I claim they are both true and at least inspired by Kant's insights: i.e. this is an updated version of Kant's philosophy of mathematics, inspired by Kant(1781)!

Some philosophical background is still available only in the above thesis. But there are many more examples and discussions on this website, some indicated below, especially the "impossible" web page, discussing abilities to detect, reason about, and (in some cases) make use of mathematical impossibilities or necessities:
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/impossible.html (also PDF)

A partial index of discussion notes in this directory is in
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/AREADME.html

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CONTENTS

What is mathematical knowledge? Hume vs Kant

In the late 1950s (around 1958-9), after a degree in mathematics and physics in CapeTown, I was studying mathematics in Oxford when I became friendly with several graduate philosophy students. When discussion turned to the nature of mathematics, the claims they made about the nature of mathematics seemed to me to be deeply mistaken.

Roughly, the philosophers I met seemed to accept something like the claim I later discovered David Hume had made, namely that there are only two kinds of knowledge ("Hume's fork"):

> (i) relations between ideas, such as explicit definitions and propositions that are derivable from definitions using pure logic (e.g. "All bachelor uncles are unmarried"), and knowledge obtained by "abstract reasoning concerning quantity or number",

and

> (ii) knowledge obtained by "experimental reasoning concerning matter of fact and existence", including knowledge derived from sensory experience, e.g. using observation and measurement, as in the empirical sciences, or knowledge gained by introspection.

Hume's advice regarding any other claim to knowledge or truth was: "Commit it then to the flames: for it can contain nothing but sophistry and illusion". I think theology was his main target, along with related metaphysical theories produced by philosophers. But I am not a Hume scholar, and I mention him merely as a backdrop to Kant's claim, originally made in reaction to Hume, that there are not two major categories of knowledge with content, but three, as explained below. (In this discussion I'll ignore interesting sub-divisions within the three categories.)

The Humean philosophers I encountered in Oxford acknowledged that -- unlike discoveries in physics, chemistry, astronomy, biology, history, or snooping on your neighbours -- mathematical discoveries were not empirical, so they concluded that all mathematical knowledge was in Hume’s first category, i.e. matters of definition and logic ("relations between ideas").

When I encountered such Humean claims about mathematics, they did not match my own experience of studying mathematics and making mathematical discoveries (as all good mathematics students do, even if their discoveries are re-discoveries). For example, while studying Euclidean geometry at school my classmates and I were often given tasks like "Find a construction that will produce a configuration ..." or "Find a proof that ...". In many cases, success involved using a physical or imagined diagram and performing (physical or imagined) operations on it. Some examples are included below.

Such mathematical discovery processes are very different from performing logical deductions from definitions and axioms.

Moreover, whereas experimenting with diagrams can provide empirical information, e.g. how long it takes to produce the diagram and whether the diagram is similar to some other diagram, in the case of mathematical reasoning with diagrams something deeper happens: we can discover *necessary truths* and *impossibilities* that are not mere logical consequences of definitions or hypothetical axioms. They are *spatial* consequences of other spatial relationships. Like logical consequences of logical relationships these spatial consequences are *necessary* consequences: counter-examples are impossible.

My own experience of making such discoveries as a mathematics student confirmed what I later discovered was Kant’s claim that besides the analytic truths there are additional necessary truths that are synthetic, and which can also be discovered to be true by non-empirical means -- using powers of human brains that are not yet understood, over two centuries later! I’ll give several examples below.

The main point of this document is that some of the kinds of mathematical discovery identified by Kant, are not yet replicated on computers, and are not explained by known brain mechanisms, in particular mechanisms based on discovering statistical correlations, and reasoning about probabilities. The well known and highly influential discoveries of ancient mathematicians were concerned with necessity and impossibility, not high or low probabilities.

Mathematical and causal cognition
In many cases the mathematical discoveries are directly related to causation: e.g. if you change the size of one angle of a triangle, e.g. by moving one of the ends of the opposite side then that necessarily causes the shape and area of the triangle to change. Adding exactly one ball to a box containing six balls, without removing any causes the box to contain seven balls. For more on this connection, including biological examples, see Chappell & Sloman (2007b).

Kant’s characterisation of mathematical truths as:
-- synthetic
-- knowable apriori
-- necessary

Immanuel Kant (1781) gave a characterisation of mathematical discoveries as *synthetic* (i.e. not composed of truths based solely on logical consequences of definitions), *non-empirical* (not derived from experience, like "Unsupported objects fall", "Apples grow in trees") and *necessary*, i.e. not only consistent with all known facts but incapable of being false. For example,

(a) it is necessarily true that two straight lines cannot bound a finite region of a plane surface, and
(b) it is necessarily true that a set A of objects in one to one correspondence with a set of five objects and a set B of objects in one to one correspondence with a set of three objects, where sets A and B contain no common object, will together form a set in one to one correspondence with every set of eight objects. In other words it is necessarily the case that 5+3=8.

Originally such discoveries were made using cognitive resources that are different from abilities to use modern symbolic logic to derive consequences from definitions, or from arbitrarily chosen axioms used to define a domain of entities.

However, Euclid's axioms were not arbitrary postulates, and (by definition of "axiom") were not derived from other axioms by logical reasoning: they were all ancient mathematical *discoveries*. Other sets of axioms discovered more recently, e.g. Tarski's axioms, have been shown to suffice to generate all, or important subsets of, Euclidean geometry. But their consequences do not include some of the interesting extensions to Euclidean geometry, e.g. *Origami* geometry (for more on Origami see Geretschlager(1995) and Wikipedia(2018), and the *neusis* construction described below.

Examples of non-definitional, non-empirical mathematical reasoning

For example: a straight (perfectly thin, perfectly straight) line segment has many (infinitely many) locations along the line. One of those locations divides the line exactly into two equal lengths. How do you know that must be true of all such line segments?

What if it is not a straight line but a curved line in a plane (flat) surface?

If a line is a *closed curve,* so that it has no ends, like a circle or ellipse, is there a point that divides it into two equal parts? The answer is: No.

Why? Because if P is a point on a closed curve L, such as a circle, or ellipse, or banana shaped curve, the portions of the line on each side of P are connected via a route that does not pass through P, so P does not divide the line into two parts. So it cannot divide the line into two equal parts.

Is it always possible to use *two* points to divide a closed curve into two equal parts?

Can a planar triangle have one side whose length is greater than the combined lengths of the other two sides, and if not why not?

If S is a sphere resting on a planar horizontal surface H, much larger than the surface of S, and P is a point on the sphere other than the point of contact with H and there are no nearby objects impeding the motion of the sphere, is it possible to roll S smoothly, without any slipping, along H until P is in contact with H? If so, how many different rolling trajectories can achieve this?

The examples in the last few paragraphs will be trivial for experienced mathematicians, but should allow non-mathematicians to have the personal experience of making a mathematical discovery, before thinking about whether, and how their discovery process could be replicated on a computer. Additional examples are below, and online at Sloman(2015-18).

An experienced mathematician can (sometimes) produce a set of *axioms* expressed in a logical formalism with some symbols referring to geometric entities, properties and relationships, and then derive *theorems* from the axioms using logic, as the great mathematician David Hilbert did when he "axiomatized" Euclidean geometry Hilbert(1899). But many lesser humans can make discoveries as the ancient mathematicians did, by thinking about spatial structures and transformations of spatial structures, and discovering *necessary connections* and *impossibilities*, without having any expertise in modern symbolic logic, and without using the Cartesian representation of geometry as a subset of arithmetic.

As far as I know, nobody has any idea what brain mechanisms make this possible and how they make it possible. E.g. how can a collection of neurons represent the fact that something is impossible, or necessarily true? How can a brain even represent the question whether something is impossible or necessarily true?

Most of the examples given above involved only straight and curved lines, points and lengths of portions of lines. Euclidean geometry also includes non-linearly extended structures, such as enclosed 2D regions and 3D volumes, as assumed by the question about a sphere rolling on a surface.

It also includes measures of lengths of straight or curved lines, measures of areas bounded by closed lines, and measures of volumes enclosed by surfaces. On a flat 2D surface a continuous closed boundary can be smooth, as in circle or ellipse, or with discontinuous changes of direction (i.e. corners), as in a triangle or square. Any mechanism explaining how human brains enable us to discover theorems in Euclidean geometry must explain how brains can represent and reason about all those shapes, and new shapes formed by combining them, and relationships between shapes.

Why do we not always need to have observed a huge variety of different examples with geometric properties in order to be able to derive consequences of those geometric properties by *reasoning* about them.

Many more examples are presented in Sloman(2015-18), and papers referred to there.

Reasoning about numerosity and numbers

Many researchers have investigated numerical competences in very young humans and in other animals. Some have attempted to find brain regions concerned with numerical competences. Various theories have been proposed about the extent to which numerical knowledge is innate. A recent survey is Siemann & Petermann(2018), though there is much older work, e.g. Piaget(1952).

My impression is that apart from Piaget and a few others, hardly any researchers in psychology, neuroscience or AI have studied analyses of number competences by Hume, Frege, Russell and other philosophers of mathematics, and as a result most research in psychology or neuroscience of mathematics, or AI modelling of mathematical competences, shows no recognition of the fact that uses of cardinal and ordinal number concepts depend crucially on understanding properties of 1-1 correspondence, i.e. bijection, in particularly that it is *necessarily* transitive and symmetric -- i.e.

exceptions are impossible. (Ordinal numbers are more complex, as explained below.)

There are some deceptive intermediate developmental states in which children and other animals show what looks like evidence of understanding cardinality in special situations, where in fact they merely use a different useful competence (e.g. template matching on small collections) that can be implemented in some brains without any general grasp of bijection.

Piaget understood this and his observations Piaget(1952) indicated that full (mathematical) understanding of bijection applied to physical objects does not develop until about the sixth year. (However all such claims are potentially subject to challenges based on new experimental setups that can reveal previously unnoticed earlier competences.)

A partial analysis of the roles of 1-1 correspondences in applications of number concepts, with some speculation about mechanisms required, was presented in Chapter 8 of Sloman 1978. (The online edition has additional notes and comments.)

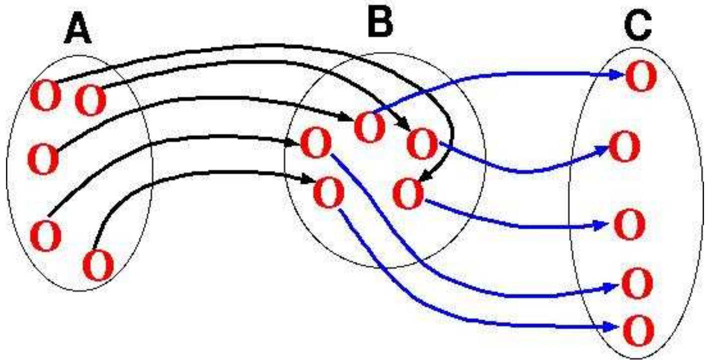
As indicated there, such correspondences are independent of sensory modes (e.g. vision, hearing, touch) and can apply not only within a sensory mode (e.g. correspondence between two visible collections) but also across domains, e.g. vision and hearing, or vision and action (where objects are moved as they are counted), or counting beads on a string while blindfold. They also apply to bijections between static configurations and temporal patterns (e.g. counting).

For now the main point is that understanding cardinal numbers requires understanding that the relation of 1-1 correspondence between two sets of items is necessarily a transitive and symmetric relation. If this were not so, many of the practical applications of number concepts would not work, as discussed in Sloman(2016). How do you know that if there are 1-1 correspondences between sets A and B, and between sets B and C then there *necessarily* exists a 1-1 between sets A and C, no matter what sorts of entities are involved?

I suggest that understanding is based on recognition that 1-1 correspondences can be concatenated, as illustrated in the figure. I.e. 1-1 correspondence is transitive.

Moreover, that is not just an empirical generalisation: someone with "normal" school-level mathematical intuition should be able to understand that the possibility of concatenating two 1-1 correspondences with a common intermediate set of objects to form a new one, is independent of the particular numbers of set elements involved, the types of elements, where they are located in space, etc. Moreover, that insight requires a kind of "schematic" spatio-temporal understanding, not reasoning from definitions using logic. It is schematic, because the particular example can be understood to have features that are independent of the number of items involved.

Fig: Transitivity of 1-1 Correspondence



This is a topological discovery, that is probably first made in connection with spatial correspondences, but is later generalised to include temporal sequences and eventually collections of abstract items (e.g. number names).

The figure illustrates only one case, yet it appears that at least some human brains, though only after several years of development, as suggested by the findings in Piaget(1952), are able to understand that the transitivity cannot have exceptions: the basis of total reliance on transitivity in many different contexts.

All this can be understood even if A and C overlap, a point that is rarely noticed in discussions of numerical cognition. E.g. there can be 1-1 correspondences between a set of boys and a set of mugs, and between the set of mugs and the set of girls. And if there’s a set of chairs in one-correspondence with the mugs, and some chairs are occupied by girls and some by boys, then both the set of boys and the set of girls is in 1-1 correspondence with the children on chairs, despite the overlaps.

For this reasons, diagrammatic explanations of numerical relationships can be misleading if no such overlaps are ever included. All this is second nature to mathematicians and most adults who are experts at counting and making uses of cardinality of sets. This is just one of many requirements that brains need to satisfy in order to support familiar but incompletely analysed competences, whose description in academic journals is often inaccurate. (I am also guilty of this.)

At what stage recognition of necessity/impossibility develops in individuals or in our evolution and what brain mechanisms and cultural support mechanisms are required are probably still unknown, although Piaget investigated some of the questions in his final two books (1981,1983). Unfortunately he lacked expertise in computational modelling, though he recognised the need shortly before his death. (In a speech at a conference in Geneva in 1980.)

As far as I know there is no known psychological or neural theory that identifies the mechanisms that support such recognition of necessary transitivity. For example Mareschal & Thomas(2006) don’t even mention the requirement despite their focus on Piaget, who certainly understood it.

Nevertheless, this is an important feature of mathematical cognition, illustrating Kant’s claim that arithmetical knowledge is synthetic, non-empirical, and necessary.

Understanding the use of numbers in continuous measures, e.g. distance, height, width, weight, etc. requires a much more sophisticated development, which may have occurred later in human history, though it builds on the evolutionary heritage underlying the uses of cardinals and ordinals, since measuring continuous quantities (e.g. length) using numerical values depends on one to one correspondences between locations (marks) on identical measuring rods, or other devices, e.g. human paces.

Deep learning mechanisms, using statistical evidence to derive probabilities are incapable of discovering, or representing, impossibility or necessity and therefore cannot acquire the kinds of knowledge described here. To that extent they are incapable of understanding the numerical concepts we, like ancient mathematicians, understand. However, it is no accident that arithmetical operations in computers give the same results as additions, subtractions, multiplication, etc. of the numbers discussed here.

That is because we have designed computers so that there is a mathematical (necessary) relationship between our arithmetic and bit-based computer arithmetic, although there are differences in speed and accuracy of execution. But a lot more has to be added to a computer system to enable it to make the discoveries made by ancient humans, such as that there are infinitely many natural numbers, and they include infinitely many prime numbers.

Note on ordinal numbers

(This section is copied from a separate document http://www.cs.bham.ac.uk/research/projects/cogaff/misc/cardinal-ordinal-numbers.html)

There is a rarely acknowledged complication regarding instances reappearing that is crucial to the difference between cardinals and ordinals. If the people who visit your office do so in the following order:

```
Andrew   Basil   Carol   Andrew   Daphne   Edmund
```

Then if there was only one person called Andrew, that person was was both first and fourth. I.e. a particular individual can occupy two locations in an ordinal structure, but not in a cardinal structure. Edmund was sixth even though only five people in total entered your room. Moreover, the question "When did Andrew visit?" has two answers: first and fourth.

Another complication allowed in ordinary parlance would be for two people to come at the same time, in which case they might both be third. Such complications could be crucial to unravelling a murder mystery, for example, but will not be pursued here.

A slightly different complication is allowed in many sporting events, which do not allow the same individual to occur in two positions in the results, but do allow two individuals to occupy the same position, with the proviso that the next position is empty. E.g. if Basil and Carol tied for second place, the results could be as followed, with nobody in third place:

```
1st: Andrew  2nd:{Basil, Carol}  3rd:  4th: Daphne   5th: Edmund
```

Yet more examples come from cyclic ordinals e.g. days of the week, months of the year, individuals sitting around a table.

These examples indicate that the concept of an ordinal structure, as used in everyday language and thought, is more complex than the concept of a cardinal.

That fact is not reflected in the mathematical theory of ordinals, as far as I am aware, since it does not allow duplicates, ties or cycles!

Testing your own understanding of 1-1 correspondence
Most mathematicians will very easily be able to answer this question, though non-mathematicians may have to think a little longer. Suppose there are two non-overlapping collections of objects C1 and C2. Is it possible for C1 to be in one-one correspondence with a *proper subset* of C2, while C2 is simultaneously in one-one correspondence with a *proper subset* of C1. If such correspondences both exist, what can you infer about C1?

This is a simplified variant of a well known theorem that some readers will recognise (The Cantor-Schröder-Bernstein theorem). When I encountered the theorem as a student I decided to try to find the proof myself. I spent much time in the following days lying on my bed with my eyes shut, until I found the proof. So much for theories of mathematical cognition as embodied. Of course, it is embodied insofar as brains are required.
[Later I may add a note with more detail about the theorem and alternate proofs, using spatial or formal/logical reasoning.]

Reasoning about areas

Ancient mathematicians discovered ways of reasoning about enclosed planar areas, for example in proving Pythagoras' theorem, which states that if ABC is a right angled triangle, with the right angle at A, then the area of the square on side BC is equal to the sum of the areas of the squares on the other two sides, AB, and AC. There are very many ways of proving that theorem. Some of them involve constructing triangles and using other theorems to prove that two triangles with a common side and a common height have the same area. However, there are also elegant proofs that involve making copies of triangles and squares and rearranging them, as in the following video demonstration (which uses a proof that is sometimes described the "Chinese proof of pythagoras theorem"):

Video proof of pythagoras theorem:
https://www.youtube.com/watch?v=0ZbB_-ip9VU
(please accept my apologies for referring to four triangles as four squares!).
For this diagrammatic proof to work do all the components have to be drawn with perfect precision? If not, how can we use imprecise diagrams and transformations to represent "perfect" Euclidean shapes and processes, as mathematicians have been doing for centuries, using drawings in sand, on slate, on paper, and various other other surfaces, and also imagined shapes, including imagined shapes indicated by a teacher pointing a bits of space, or tracing imaginary lines through space. All of these can play essential roles in mathematical discovery and mathematical communication.

How is that possible, given that similar techniques can't be used to prove generalisations in physics, chemistry, geology, biology, etc.?
Or a more detailed presentation here by Eddie Woo:
https://www.youtube.com/watch?v=tTHhBE5lYTg

Note the use of human abilities to perceive, manipulate and reason about spatial relationships, as opposed to logical or algebraic formulae.

In contrast, the demonstration based on allowing water to flow from two small square containers to a larger square container, in the following video, does not present a proof. Why not?
https://www.youtube.com/watch?v=CAkMUdeB06o

Computer-based geometry theorem provers

Examples of early automated geometrical reasoners by Gelernter and Goldstein were referenced in Note [+] above. However, automated theorem provers have developed enormously since then, and there are now far more advanced geometry theorem provers, some reported in Ida and Fleuriot(2012)

Computer-based AI reasoners of that general sort are able to derive theorems in Euclidean geometry by constructing (and checking) proofs based on modern, logical, formulations of Euclid's axioms and postulates (e.g. Hilbert's or Tarski's axiomatisation), but they cannot replicate the original discovery processes based on mathematical *intuition* (using still unknown cognitive mechanisms in brains), that somehow enabled ancient mathematicians to discover Euclid's axioms, and centuries later Hilbert's and Tarski's axioms (among others).

If the Euclidean theorems are all stated in this form
IF then
where is a conjunction of all the axioms and postulates in Euclid's *Elements*, then exactly what the status of such a theorem is, and what the status of is depends on what is in the axioms.

If the axioms are all expressions in standard logical (e.g. predicate calculus) notation, along with some abbreviative definitions, and the theorems are provable using only logically valid inferences (whose validity depends only on the logical forms used, not the contents referred to) then in a "modern" interpretation of Kant's ideas, the consequents are all analytic, as opposed to synthetic conclusions which require either additional axioms, or some form of reasoning that is not purely logical, but makes use of insights into properties and relations of spatial structures, for example.

Moreover there are geometrical discoveries that are not derivable from Euclidean geometry, e.g. the neusis construction explained in
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/trisect.html and Mary Pardoe's construction, described in
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/triangle-sum.html, which supports a proof of the triangle sum theorem without reference to parallel lines. I suggest that the brain mechanisms required for the ancient mathematical discoveries are related to Immanuel Kant's claims about mathematical knowledge as being non-empirical, non-analytic and non-contingent, alternatively expressed as *a priori*, synthetic and necessary, as explained in Sloman(1965).

Brain mechanisms required for those ancient discoveries are still unknown. It may turn out that they depend essentially on the mixture of discrete and continuous molecular processes *inside* synapses rather than being explicable in terms of signals passing *between* neurons. Some half-baked ideas about this are being explored elsewhere.

Discovery/invention of differential/integral calculus an example?

This section is mainly for readers who are already acquainted with differential and integral calculus, at least at a fairly low undergraduate level, with personal experience of solving typical student problems (finding derivatives or integrals).

A very useful "short history" of the development of "infinitesimal calculus" can be found on Wikipedia:
https://en.wikipedia.org/wiki/History_of_calculus
A possibly useful supplement that I have not yet examined closely is
https://medium.com/explore-artificial-intelligence/the-birth-of-calculus-8e14e01f4550

I shall later return to this paper and expand on the following claim: the various mathematical discovery steps leading up to and including (for example) the achievements of Leibniz and Newton, including the mathematical discoveries and the uses of those discoveries in explaining and predicting physical phenomena (e.g. astronomical observations, and tidal phenomena), could not be replicated by any of the mechanisms developed in AI since it began in the work of Turing and others, despite the fact that there have been computer programs, designed by humans as opposed to being produced by machine learning systems, that solve various subsets of those problems.

Whether some future advance in AI will falsify this claim is a separate question, as is the question whether some fundamental new design for computers that more closely represents mechanisms in brains (e.g. sub-neural chemical mechanisms) will be required to support such discoveries.

-----WHEN I RETURN TO THIS PAPER I'LL REMOVE THESE TWO LINES--------
-----ITEMS BELOW ARE PLACE-HOLDERS FOR SECTIONS TO BE ADDED------

Evolution's use of compositionality (and Kant)

The term "compositionality" is most often used to refer to features of language involving at least two types of structure with systematic relationships between them, in particular:
-- *syntactic* (grammatical) structures of phrases, sentences, paragraphs, etc.
-- *semantic* structures expressed or denoted by those linguistic items
However it is useful/illuminating to point out that in a generalised sense "compositionality" is a feature of many aspects of biological evolution and its products, including its information processing mechanisms. For more details see Sloman(2018c), which explains, among other things, how the uses of compositionality in evolution, and in individual development as genomes are expressed, can involve mathematical structures and processes.

I think it is helpful to see Kant's philosophy of mathematics as an early attempt, based on remarkably deep insights, to describe and explain some of the most important features of ancient human mathematical discoveries and the mechanisms that made those discoveries possible.

Did Lakatos refute Kant?

One of the fundamental requirements for mathematical thinking is being able to organise collections of possibilities and making sure that you have checked them all. If you can't do that you don't have a mathematical result, only a guess.

How can you know that you have checked all possibilities? The history of mathematics shows that even brilliant mathematicians can make mistakes Lakatos (1976). This means that the traditional emphasis on the role of "certainty" in mathematics may be misguided: *certainty*, or its absence, like *infallibility* or its absence, is a matter of the psychology of mathematicians, not the subject matter they investigate, which is something richer and deeper: a feature of the universe that was playing a role in evolution (the "Blind Mathematician") long before human mathematicians existed.

Computers, like drawings in sand, slates and chalk, pen and paper, 3-D models made of wires and beads, and other aids to thinking and communication, have expanded what human mathematicians can do, but not changed the nature of the subject matter. Some are tempted to conclude that mathematics is essentially a social phenomenon. That may be true for relatively weak mathematicians, though there are others who do their main work struggling with problems, not talking to colleagues.

Nowadays the role of colleagues is increasingly being supplemented by various roles of computers in supporting mathematical research, some discussed in Wolfram(2007).

However, Kant's ideas about the nature of mathematical discovery, and roles for mathematical insight or intuition in making discoveries, remain relevant to many human mathematical discoveries, even if there is increasing use of computers to aid mathematical research, including use of logic.

Some themes to be added

Work is in progress on a theory relating processs of biological evolution and mechanisms by which information acquired during evolution can be used in descendents of the first users,

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