What Sort of Information-Processing Machinery Could Ancient Geometers Have Used?

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Abstract. Automated geometry theorem provers start with logic-based formulations of Euclid's axioms and postulates, and often assume the Cartesian coordinate representation of geometry. That is not how the ancient mathematicians started: for them the axioms and postulates were deep *discoveries*, not *arbitrary postulates*. What sorts of reasoning machinery could the ancient mathematicians, and other intelligent species (e.g. crows and squirrels), have used for spatial reasoning? *"Diagrams in minds"* perhaps? How did natural selection produce such machinery? Which components are shared with other intelligent species? Does the machinery exist at or before birth in humans, and if not how and when does it develop? How are such machines implemented in brains? Could they be implemented as virtual machines on digital computers, and if not what human engineered "Super Turing" mechanisms could replicate what brains do? How are they specified in a genome? Turing's work on chemical morphogenesis, published shortly before he died suggested to me that he might have been considering such questions. Could deep new answers vindicate Kant's claim in 1781 that at least some mathematical knowledge is non-empirical, synthetic and necessary? Discussions of mechanisms of consciousness should include ancient mathematical diagrammatic reasoning, and related aspects of everyday intelligence, usually ignored in AI, neuroscience and most discussions of consciousness.

Keywords: Geometrical/topological reasoning - Evolution - Kant - Turing - AI

1 Introduction

Some theories of consciousness make use of mathematics, e.g. mathematical models of neural processes, but no theory that I have encountered explains how brains enable great mathematical discoveries to be made, e.g. the deep discoveries in geometry and topology, made many centuries ago, some of which, in Euclid's *Elements*, are still in regular use world-wide.[1] AI geometry theorem provers since the 1960s start with logical formulations of Euclid's axioms, whereas for ancient

[1] A 16 page paper introducing aspects of the Turing-inspired Meta-Morphogenesis project http://goo.gl/9eN8Ks submitted to the 2018 *Diagrams* conference, was accepted as a short paper. The original version is at http://goo.gl/39DRCT

mathematicians the axioms and postulates were not arbitrarily chosen starting points but deep *discoveries*, selected as “axioms” because other geometrical facts could be derived from them, even if originally discovered independently. Moreover, such mathematical discoveries concern necessary truths and impossibilities, which are not discoverable (or even representable) by statistics-based learning mechanisms. Necessity is not extreme probability. However, it is important not to confuse the necessity/impossibility in the *content* of mathematical discoveries with any claim that human mathematical reasoning is infallible. Many mathematicians have made mistakes that were later corrected by mathematical reasoning, sometimes triggered by empirically discovered counter examples. (However, the forms of consciousness involved in those discoveries seem to have been ignored by philosophers and scientists studying consciousness in recent decades.)

Not all geometrical reasoning is based on Euclid’s axioms. Standard proofs that angles of a triangle sum to 180°use Euclid’s parallel postulate, but around 1970 Mary Pardoe discovered, while teaching school mathematics, that it can be proved without using parallel lines, by considering an arrow lying on one side of the triangle then rotated in turn through each (internal) angle of the triangle. It must end up on the initial side pointing in the opposite direction, after turns totalling half a rotation, as shown in Fig. 1[2] What brain mechanisms allow such discoveries to be made and understood? As far as I know, nothing in current neuroscience or in current AI explains such discovery capabilities.

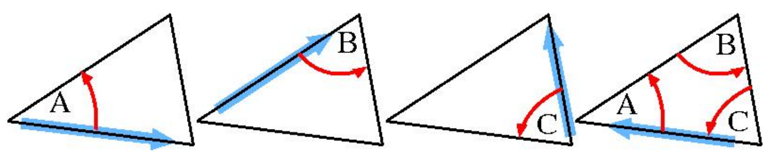

Fig. 1. Mary Pardoe’s proof of the triangle sum theorem. Her pupils understood and remembered this more easily than the standard proof, using parallel lines. See http://www.cs.bham.ac.uk/research/projects/cogaff/misc/triangle-sum.html

Many important geometrical discoveries can be made without starting from Euclid’s axioms. For example, Origami techniques allow forms of reasoning that go beyond what is provable in Euclidean geometry. Extensions of Euclidean geometry include the *Neusis* construction, known to ancient mathematicians, but not included in Euclid’s *Elements*. It involves use of a movable straight edge with two marks, and allows arbitrary angles to be trisected easily.[3] The discovery of non-euclidean geometries was another important example, famously used by Einstein in his General Theory of Relativity.

Topological reasoning seems to be even more widespread, as discussed in [1]. Young children who have never studied logic or algebra can tell that it is

[2] See http://www.cs.bham.ac.uk/research/projects/cogaff/misc/triangle-sum.html

[3] http://www.cs.bham.ac.uk/research/projects/cogaff/misc/trisect.html

impossible for two linked rings made of solid, impermeable matter to become unlinked without at least one of them changing shape (e.g. ceasing to be a ring). This can be seen in their responses to clever stage magicians who make it *look* as if the impossible has been achieved. What brain mechanisms enable us to see that such things are impossible?

Some researchers seem to believe that given appropriate training, deep learning mechanisms could replicate all ancient geometrical discoveries. But statistics-based mechanisms and can only discover that certain generalisations have high, or low, probabilities. They cannot discover *necessities* and *impossibilities*, as Kant[2] showed when he pointed out gaps in Hume's classification of types of knowledge. Neural nets cannot even *express the idea* of something being impossible, or necessarily the case. Kant argued that there are important types of *non-empirical* mathematical knowledge about *necessary* truths and *impossibilities*, for which statistical evidence can never suffice.[4] What enables humans to understand these concepts, if neural nets cannot express necessity or impossibility? Is there a spatial configuration in which a planar triangle and a circle have have exactly seven boundary points in common? You can do mental experiments with imagined triangles and circles to answer this, unlike AI systems that use Hilbert's axiomatisation of geometry, and cartesian coordinates, to answer such questions, unknown to ancient mathematicians. Cartesian coordinates were not discovered until centuries later. Can any current AI system replicate that discovery?

2 Why Is Non-empirical Knowledge of Non-contingent Truths Important?

The kind of mathematical knowledge under discussion, is not just a philosophical oddity. It is of great practical importance to intelligent agents. Knowledge of impossibility makes it possible to rule things out without testing. Likewise knowing that having one feature of an object or process necessarily implies another allows complex decisions to be taken and used with confidence, including choice of routes in a cluttered environment, and many others.

Not only humans benefit from this kind of reasoning. Evolution has used many mathematical discoveries in selecting both physical or chemical structures and control mechanisms for those structures. Negative feedback control is used in many "homeostatic" control mechanisms from the very simplest organisms to control of blood pressure, temperature, chemical balances etc., in complex organisms. This required evolved construction kits with mathematical properties.[5]

A particular example of non-spatial mathematical intelligence in young humans is the ability to create subsuming generative grammars after many patterns of verbal communication have been found to work in the environment. This has

[4] However, modal operators, e.g. "necessary", "impossible" should be analysed using "possible configuration" not "possible world" semantics.

[5] Some speculations about evolved construction kits are online here: http://www.cs.bham.ac.uk/research/projects/cogaff/misc/construction-kits.html

the great benefit of allowing novel linguistic structures to be created, or to be understood, without first learning them all from examples.

In language development, this process is followed by a further level of competence in adjusting the mechanisms to cope with exceptions to the grammatical rules. That is a rather messy kind of mathematical process. Unlike other forms of mathematical reasoning, the ability to derive new linguistic utterances to communicate novel thoughts is not guaranteed to be successful because of its dependence on the competences and vagaries of other humans.

3 Meta-level Competences

A creative engineer requires additional layers of competence: meta-meta- knowledge about how to search spaces of mathematical structures to find new techniques when faced with novel problems. I don't claim that evolution produces built-in knowledge of all the kinds of mathematical knowledge used by humans: some are products of individual discovery or cooperative cultural evolution, including full understanding of cardinal numbers, which requires understanding that one-one correspondence is a transitive and symmetric relation, which Piaget's work suggests does not develop in young humans for five or six years.[3]

I suspect various types of mathematical development are special cases of *staggered* gene expression: over time, brains develop new layers of meta-competence that evolved later than others, and which provide new forms of learning/discovery applicable to products of layers that evolved earlier and develop earlier in individuals, as suggested crudely in Fig 2, allowing greater developmental leaps across generations, based on a "Meta-configured" genome. (This is very different from fashionable deep learning mechanisms.)

Individual multi-layered development seems to depend on the features of genome expression in intelligent animals summarised (roughly) in Fig 2,[6] Recently developed genetic abstractions from previously evolved competences can be instantiated in novel ways in each generation, illustrated crudely in the figure, allowing greater developmental variety in products of a shared genome, including greater leaps across generations than could be achieved by a fixed learning mechanism provided by the genome. The history of human uses of various types of diagram seems to provide examples of this mechanism.

Current AI, including logic-based reasoning mechanisms (argued by McCarthy and Hayes to be adequate for intelligent systems [5]) and the fashionable "brain-inspired" mechanisms based on statistical learning, e.g. those surveyed by Schmidhuber in [6], cannot match the *spatial* insight-ful reasoning capabilities produced by these mechanisms. Current neural models deal with networks of nodes with numerical attributes and linked numerical relationships, whereas for the kinds of mathematical discovery I am discussing it is not necessary to collect statistical data from samples. E.g. mathematicians often reason using spatial manipulations of represented spatial structures: "diagrams in the mind"[7].

[6] goo.gl/3N1yQV gives more detail (still expanding).

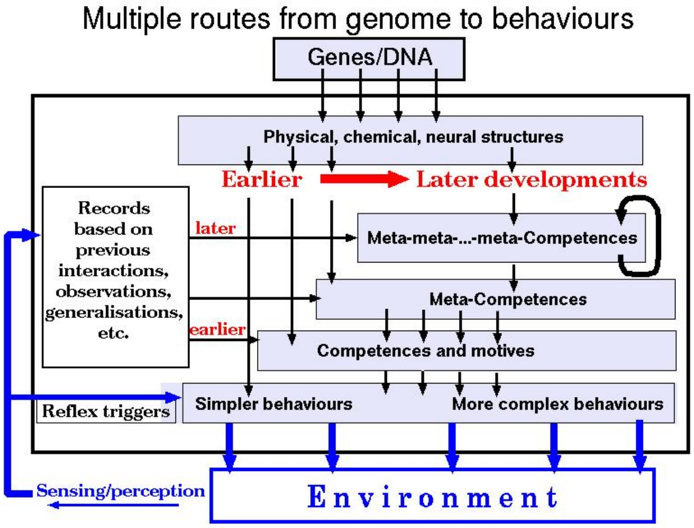

Fig. 2. Staggered "waves of expression" of the Meta-Configured Genome: lower layers begin development earliest via genetic influences crudely depicted on the left. Processes further to the right and higher up occur later, building on records of earlier processes that help to *instantiate* more recently evolved genetic abstractions that are expressed later in development, including new motive-generators. (Based on [4].)

Perception and use of spatial affordances, by humans and other animals acting in natural environments, require abilities to perceive and reason about spatial structures and spatial relationships, including topological relationships such as containment and overlap, and partial orderings (nearer, wider, more curved, etc), rather than precise measures [8].

4 Back to Ancient Mathematical Reasoning/Discovery

By examining examples of the spatial (diagrammatic) reasoning involved in ancient mathematical discoveries we may hope to gain some insights into what is missing from current forms of computation. An example that has a number of interesting features, including very easy comprehension by non-mathematicians is looking at a configuration of cup and spoon on a saucer and thinking about how to get the saucer and spoon onto the cup, using only one hand.

Similar points could be made about various stages of nest construction by birds, e.g. weaver birds,[7] that require abilities to perceive structures, select items to manipulate, moving them to new required locations, and then taking actions to enable the new items to be part of a growing stable structure.

Conjecture: Information processing mechanisms required for practical purposes in structured environments evolved in many species, mainly using reason-

[7] Illustrated by the BBC here https://www.youtube.com/watch?v=6svAIgEnFvw

ing about topological structures and relationships and partial orderings (e.g. of distance, size, speed, angle, etc.) rather than *metrical* information. In humans, the mechanisms were used in new ways, in conjunction with new meta-cognitive and meta-meta-cognitive mechanisms, leading eventually to explicit mathematical reasoning, discussion, and teaching, about topological and geometrical aspects of structures and processes in the environment.

As organisms evolve to cope with more complex structures and processes *in the environment*, they use increasingly complex abilities to create and manipulate new *internal* information structures, representing parts and relationships of external structures and processes, and supporting reasoning about consequences of possible actions, as hypothesised by Craik in 1943 [9]. Later, newly evolved meta-cognitive mechanisms, for reflecting on and comparing successes and failures of such reasoning processes, allowed new, mathematical, aspects of the structures and relationships to be discovered, thought about, and, in some cultures, communicated and used in explicit teaching and discussion. Much later, via social and cultural processes for which I suspect historical records are not available, the materials came to be organised systematically, recorded in various external "documents", such as Euclid's *Elements* and taught in specialised sub-communities.

If, as I suspect, understanding of cardinality depends on such mechanisms, then psychological evidence purporting to show innate understanding of cardinality, shows nothing of the kind: only that there are some simpler pattern recognition abilities that give observers the illusion that young children or other animals understand cardinality.

5 Towards a Super-Turing Geometric Reasoner

There are deep, largely unnoticed, aspects of the ways human and non-human animal minds work that are closely connected with the mechanisms underlying important non-numerical mathematical discoveries by ancient mathematicians, i.e. topological and geometrical discoveries. For ancient mathematicians the axioms and postulates in Euclidean geometry were not arbitrarily chosen starting formulae from which conclusions were derived: the axioms were all major *discoveries*, using mechanisms still available to us. And they did not use the arithmetisation of Geometry based on Cartesian coordinates.

What mechanisms allow you to discover what happens to angles of a triangle as it gets stretched by motion of one vertex relative to the other two. E.g. what will happen to planar triangle ABC, such as the triangle depicted in Fig.3, if vertex A continually moves further from the opposite side, BC, along a line through A that intersects BC, as illustrated in Fig. 3. Even non-mathematicians can work out that as A moves further from BC the angle BAC will steadily decrease, without knowing exact lengths of lines and sizes of angles. Despite being so obvious to non-mathematicians, this answer has surprising mathematical sophistication. It involves both the continuum of locations of the angle A, and the continuum of sizes for the angle A, and a systematic relationship between the two continua: as the distance increases the angle size decreases. It is not obvious

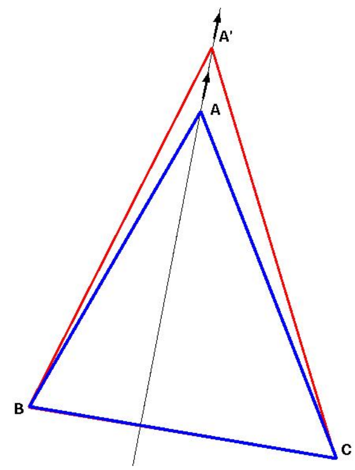

Fig. 3. How does the angle at A change as A moves further from BC, along a straight line that passes between B and C, e.g. moving to A′? What brain mechanisms allow reasoning about such questions?

exactly how the angle size and the length are related, though it is obvious that as one increases the other decreases, unless the line along which A moves intersects the line through B and C outside the segment BC.[8] What mechanisms would enable a future robot to find the relationships in Fig. 3 as obvious as we do? Brains seem to do much that is not explained by current neural net mechanisms nor by current AI models of spatial reasoning using logic or logic plus algebra, trigonometry etc. How might the mechanisms differ from a Turing machine,with its linearly ordered tape, divided into locations each of which can contain exactly one symbol?

6 Conclusion

A challenging research problem is to find a way to specify a type of machine that could replace a Turing machine's tape, tape-head, and symbol table, with something like a membrane on which marks can be made and which can be stretched, rotated, translated, and its new position compared with the old position, to see what has changed, with at least two layers of meta-cognition detecting and reasoning about what does and does not, and what can and cannot change. Humans thinking about the triangle problem seem to construct imagined states on which very much more complex operations can be performed, including two or

[8] The case where A moves along a line that intersects BC outside the triangle is discussed in another document. See
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/apollonius.html.
Surprising additional complexities are discussed in that and
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/deform-triangle.html

more co-ordinated continuous changes, and two or more levels of meta-cognition operating in parallel: e.g. one detecting and summarising changes, and another reasoning about the nature of those changes – e.g. discovering necessities and impossibilities. Is there a minimal set of basic mechanisms (perhaps chemical mechanisms in brains?) from which all the forms of spatial reasoning required for an intelligent animal can be derived? [9]

The Meta-Configured genome hypothesis sketched above implies that intelligent animals do not have a uniform innate learning mechanism that operates from birth on increasingly complex and varied data sets. Different mechanisms, with different evolutionary origins modified to fit the individual's environment, come into operation at different stages of development over an extended time period. Compare language development and Karmilof-Smith's ideas about "Representational Re-description"[10].

I suspect Alan Turing may have been working on a problem of this sort when he wrote "The chemical basis of morphogenesis" [11], now his most cited paper. What would he have done if he had not died two years after it was published?

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[9] Later developments of the idea of a Super-Turing machine will be added here: http://www.cs.bham.ac.uk/research/projects/cogaff/misc/super-turing-geom.html