Biological/Evolutionary Foundations of Mathematics (BEFM)

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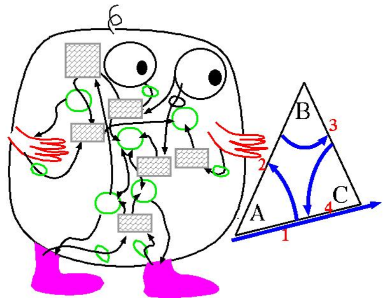


How can evolution discover mathematical structures?
How can evolution parametrize mathematical structures?
How can evolution produce babies that grow up to be mathematicians?
Can we produce baby robots that grow up to be mathematicians? How?
What mechanisms are required? What mechanisms did evolution use, and produce?
What must the physical universe be like to make all this possible?

PREAMBLE:
This is not a report on a finished project. It is a partial progress report on a large scale multi-disciplinary collaborative project. At present there's a great shortage of collaborators! The project links foundations of mathematics to the foundations of life, mind and mathematical thinking – and also to cosmology.

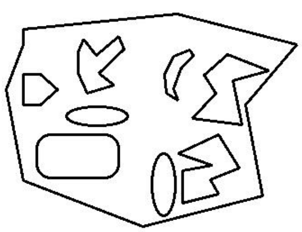
There are two main themes arising out of two related questions: (1) what are the properties of the *fundamental construction kit* (FCK) that makes possible formation of galaxies, planets and life-supporting environments, and also provides mechanisms used by life forms of all types, including an enormous variety of information-processing capabilities serving many different biological functions, and (2) what are the actual *evolutionary and developmental trajectories* that account for the variety of life forms that have already come into existence and their achievements, especially the achievements involving discovery and use of various kinds of mathematics.

Both themes raise mathematical questions (including meta-mathematical questions):
(1) Can we give a mathematical characterisation of the features of the FCK that can be demonstrated to be adequate to support all the known forms of life – including all the known varieties of biological information processing? (Compare questions about the sufficiency of a set of axioms in a formal system intended to characterise some domain.)
(2) Can we identify the processes (including the evolutionary and developmental trajectories) by which evolution produced particular kinds of organism, explaining not only their physical forms, but also the information processing capabilities involved in their development, learning, decision making and acting. (Compare trying to find a proof of some formula in an axiom system.)

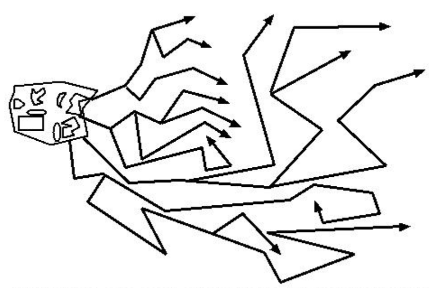
Those are very general questions but for this discussion I am specially interested in how the FCK supports evolution and development of mathematical capabilities, and the processes by which various sorts of mathematical potential are realised, including the use of mathematical structures in processes of reproduction and development, and later on the ability of some organisms to acquire and use mathematical knowledge of increasingly sophisticated forms, some of them including mathematical meta-knowledge: mathematical knowledge about mathematics.

Theme (1) is about the generative powers that existed in the FCK long before those powers were manifested in multiple strands in the evolution of the universe, including powers involved in forms of life on our planet. Those powers include the ability (eventually) to support many types of information processing required for or produced by life forms. Some of those information proccessing abilities were required at very early stages in evolution, serving functions such as reproduction, sensing, control of simple behaviours, and metabolism. Some of the derived powers first evolved billions of years after life began, including sophisticated mathematical reasoning capabilities, such as reasoning about continuous deformatinons of spatial structures e.g. peeling fruit, extracting food from a carcass, putting on clothing, making or undoing knots, and many more. Moreover, the initial generative powers in the FCK extend far beyond what they have produced at any time, as witnessed by continued production of novel achievements in mathematics, the empirical sciences, technology, philosophy and various art forms.[1] I think it is fair to say that at present nobody has a complete characterisation of the features of the initial toolkit that make all this possible, though it is clear that chemistry and quantum physics play important roles.

Theme (2) is about the *multi-layered* processes in which the generative potential of the FCK can be and has been realised, especially in the use by evolution of increasingly abstract and powerful mathematical structures. The processes are multi-layered in the sense that it was impossible for natural selection and its products to go direct to some of the more complex designs without the use of intermediate designs for organisms that form part of the evolutionary history of the more complex designs. Even the processes of evolution had to evolve, for example through use of sexual reproduction which did not exist initially, use of cognitive processes in mate selection, and use of various kinds of symbiotic relationship.



The Fundamental Construction Kit (FCK)



Evolutionary and other trajectories from the FCK through the space of possible designs (very much over-simplified)

Visible and invisible design features

Several past thinkers (e.g. Goethe, D'Arcy Thompson, Goodwin, Turing, and others[2]) have studied examples of mathematical structure in physical products of biological evolution, and in the processes of physical development producing those structures. They focused on attempting to describe and explain *observable* and *measurable* products of evolution and development, for instance types of physical shape and physical motion, including the mathematical patterns instantiated – such as the patterns observed in motion of quadrupeds moving at different speeds.[3]

In contrast, the main focus of the project outlined here is the history of mostly *invisible* mathematical structures and processes that have been, or are, involved in increasingly complex

[1] This is perhaps the most extreme generalisation of the ideas about "actual possibilities" presented in (Sloman, 1996).
[2] For example, see (Turing, 1952), (Boden, 2006, Sections 15x(b-d), Vol 2), (Lambert, Chetland, & Millar, 2013)
[3] Illustrated here: `http://en.wikipedia.org/wiki/Horse_gait`

forms of *biological information processing*, in reproduction, development, learning, and other mental and social processes. Eventually those included proto-mathematical discoveries in organisms that perceive and make use of affordances (Gibson, 1979; Sloman, 2011b), followed later by mathematical discovery and reasoning by humans (as reported in Euclid's *Elements*, for example, and, in the last few decades, computer-based automated discovery and reasoning machinery. For reasons that don't seem to be widely recognized, or well understood, that automated reasoning machinery does not yet match human spatial reasoning (Sloman, 2014).

We can construe all of the processes involved in formation of life-supporting galaxies, planets, chemical soups, then biological evolution, through microbes, many forms of plant and animal life, humans, social systems and ecosystems as being (in part) mathematical phenomena, since they exemplify mathematical structures of many kinds (e.g. feedback control loops in microbes, various kinds of symmetry, grammars, etc.). More precisely they all involve multiple interacting instances of mathematical structures – and such things were already going on long before there were humans. Eventually, the processes included humans discovering and presenting mathematical theorems, proofs of theorems, and in some cases applications of those theorems in practical activities, e.g. constructing complex buildings.

Discovering as much as possible about those evlutionary, developmental, social, and technological trajectories is a central aim of the Meta-Morphogenesis project.

Note on information

The notion of "information" used here (and in much of science and engineering) is not Shannon's notion (Shannon, 1948; Ritchie, 1986) but the older notion, familiar to Jane Austen,[4] which refers not to a numerical measure, nor to bit patterns or other information bearers, but to the contents of questions, intentions, commands, conditional tests, thoughts, hopes, fears, conjectures, beliefs, percepts and assertions. (An extended, but partial, analysis of this concept is presented in (Sloman, 2011a), where it is claimed that no explicit, non-circular, definition of "information" in this sense is possible. That's also true of other "basic" concepts such as "matter" and "energy".)

Some unanswered questions

As the preceding discussion indicates, the project outlined here addresses these "Kantian" questions, which are close in spirit to questions raised in (Kant, 1781):

Q1 What features of the universe initially made possible the evolutionary processes that produced all the mechanisms and successive mathematical discoveries implicit in evolutionary designs or explicit in the thinking of individual organisms? (Compare Theme (1), above.)

Q2 What makes it possible for increasingly sophisticated mathematical capabilities to be produced by evolution? (Compare Theme (2), above.)

An answer to Q1 must be some sort of generic specification of the conjectured initial "construction-kit", available before life began, plus a demonstration that whatever meets that specification will have the generative power required to support the layers of evolution and development described in the answer to Q2.

An answer to Q2 should be a collection of descriptions of biological transitions, including branching transitions, e.g. in evolution of new genomes, in types of individual learning and development based on various genomes, in learning, in cultural evolution, in ecosystems, etc., where all the transitions can be shown to be made possible by the features of the initial construction kit.

[4] http://www.cs.bham.ac.uk/research/projects/cogaff/misc/austen-info.html

(Dennett, 1996) presents some example transitions, but at a high level of abstraction, and without any reference (as far as I recall) to the requirements for a construction kit of the sort that could answer Q1, except for the requirement that it should provide material for natural selection to operate on.

In other words, an answer to Q1 is partly analogous to a specification of a formal language with a set of axioms and inference rules, with a very rich set collection of increasingly complex theorems, or a generative grammar with a very rich set of increasingly complex sentences that accord with the grammar.

And an answer to Q2 is partly analogous to branching chains of proofs shown to be valid in according to the axiomatic system, which prove increasingly complex theorems, or branching chains of derivations from the grammar producing increasingly complex and varied sentences conforming to the grammar.

Answering Q1 specifies *a system* with great generative power. Answering Q2 demonstrates important *derivations within the system*. Some of those derivations will have to present evolutionary, developmental, learning and social/cultural trajectories that include transitions of the sorts that led to the mathematical discoveries in Euclid's *Elements*, among many others.

Most of the important details of that evolutionary history are not yet known, so for the time being we have to rely on intelligent guess-work, constrained by biological evidence. For example, I guess that evolution of abilities to detect and use what Gibson called positive and negative"affordances" in complex structures was a precursor to evolution of abilities to discover proofs in Euclidean geometry (Sloman, 2011b).

Examples of related work

Many thinkers have attempted to produce answers, or partial answers to Q2, presenting informed speculations (or guesses) about the evolutionary history of human intelligence and other forms. These are all concerned with evolution of forms of biological information processing rather than evolution of physical features or behaviours. Examples include (Dennett, 1996), (Donald, 2001) (Gärdenfors, 2003) Related questions, about the "triangle" of explanatory relations linking *matter*, *mind* and *mathematics* are discussed in (Hut, Alford, & Tegmark, 2006), building on and criticising ideas in (Penrose, 1994). The authors don't explicitly consider the possibility of studying the topic in greater depth by examining detailed requirements for evolution of minds and mathematical capabilities, using what we have learnt in recent decades about possibilities and problems in designing machines with minds of various sorts, or at least reasoning capabilities of various sorts, by analogy with the attempt in (Ganti, 2003) to explain the possibility of primitive forms of life in terms of what was known about physics and chemistry a few decades ago.

Until very recently, scientists and philosophers have been ill-equipped for the study of information-processing mechanisms. Only after decades of increasingly sophisticated information engineering, following the development and use of electronic computers in the mid 20th century, have many of the important concepts and problems been understood, such as the concept of a running virtual machine (explained in (Sloman, 2010)), the idea of *virtual-machine functionalism* (Sloman, 2013a)[5] though we are still not ready to answer all of the questions raised here, e.g. what sorts of information-processing mechanisms can support mathematical reasoning about constraints on continuous shape deformation.[6]

[5] `http://www.cs.bham.ac.uk/research/projects/cogaff/misc/vm-functionalism.html`, and possibilities for self-monitoring in various sorts of virtual machines. Unfortunately the education of philosophers does not yet include these ideas as a matter of course, although the need was pointed out several decades ago.

[6] As in: `http://www.cs.bham.ac.uk/research/projects/cogaff/misc/torus.html` and

Answering Q2

Part of the answer to **Q2** seems to be: repeated construction of new foundations from old, over billions of years of biological evolution – initially producing very slow changes in biological uses of mathematical structures and constraints. Later, discoveries were accelerated, when mathematical discovery capabilities in the processes of natural selection were later enhanced by evolved capabilities for mathematical discoveries in *individuals*, including probably animals building nests or shelters, hunting, dismembering carcasses or edible vegetable matter, and later followed (at least in humans) by evolved capabilities for meta-cognition ("reflective" capabilities). This made possible (among other things) the ability to notice differences between empirical discoveries which required learners to interact with and study the environment, and non-empirical discoveries that could be triggered by experience but could be argued for independently of evidence from experience.

These processes were later further enhanced by new developmental and social/cultural transitions.

The conjectures presented here (and expanded below) are highly schematic: there are many details to be filled in by long term multi-disciplinary research, an important part of the Meta-Morphogenesis project, proposed in (Sloman, 2013b). So the answer to **Q2** is essentially a history of the evolution of mathematical capabilities in living things, in which layer after layer of mathematics (including meta-mathematics) was discovered and explored, a history that started long before humans existed and is still in progress.

This turns Wittgenstein's suggestion (Wittgenstein, 1978) that mathematics is an "anthropological phenomenon" on its head. Rather, the evolution of organisms like humans, using many mathematical structures, including languages with infinite generative power for thinking and communicating, is partly a "mathematical phenomenon", with a long history since times before there were any humans. Not only Wittgenstein: anyone who thinks of mathematics as a human creation has missed many roles of mathematical structures and relationships in living things that evolved long before humans. Even a virus whose reproduction uses symmetry constraints rather than explicit instructions in the genome, is a mathematical phenomenon.

Answering Q1

Part of the answer to **Q1** seems to be that all those developments were made possible, long before they actually occurred, by the fact that the physical universe provided a very powerful *fundamental construction kit* (FCK) some of whose "generative" potential was realised on this planet billions of years later, though the potential existed long before it was realised.

Compare the arithmetic competence of a child, whose understanding of counting covers far more numbers than the child has ever named. This is also related to Chomsky's distinction between linguistic *competence* (infinite in a normal language-user) and linguistic *performance* (always finite, and also includes errors of production) (Chomsky, 1965).

A feature of this answer is that it implies that before life (or proto-life, or life-enabling molecular structures) existed, the physical universe must already have provided a *construction kit*, the FCK, with the potential for assembly of increasingly complex structures and processes, including production of new kinds of construction kit used at later stages by evolution, or by products of evolution. (Some of those transitions, including conjectured transitions, are listed below.)

A more complete answer to **Q1** will require development of a theory of types of construction kit and their generative powers, including not only kits that support construction of physical structures

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http://www.cs.bham.ac.uk/research/projects/cogaff/misc/shirt.html (e.g. Lego bricks, Meccano), but also kits that support construction of information-processing machines, and kits that support construction of new construction kits with new powers (e.g. Turing machines, von Neumann machines, and more recently extendable computing networks).[7]

Inadequate construction kits

Not all conceivable physical universes would provide sufficiently rich construction-kits. For example, consider a universe composed of Newtonian particles, with masses, locations, velocities, accelerations, and various degrees of elastic compressibility, flying around in a Euclidean space, subject only to the forces of gravitational attraction and forces produced by impact (according to Newton's third law), and without adhesive forces or chemical composition. Such a universe, with simple physical structures and processes, lacking anything like chemical bonds, would not be able to grow the kinds of relatively stable, but changeable physical structures with more or less rigid parts found in our universe, especially in living things. Neither would such a universe support information-processing machinery of the sorts described below.

Newton seems to have understood some of this. He was very interested in chemistry/alchemy, asked some deep questions and formulated some interesting hypotheses.[8] In particular, he wrote (with quaint inductive reasoning): *"Have not the small Particles of Bodies certain Powers, Virtues, or Forces by which they act at a distance, not only upon the Rays of Light for reflecting, refracting and inflecting them, but also upon one another for producing a great part of the Phaenomena of Nature? For it's well know that Bodies act one upon another by the Attractions of Gravity, Magnetism and Electricity; and these Instances shew the Tenor and Course of Nature, and make it not improbable but that there may be more attractive Powers than these. For Nature is very consonant and conformable to her self."*[9]

I suppose we shall never know whether Newton would still have thought Nature "very consonant and conformant to her self" if he had learnt about modern physics, including general relativity and quantum physics.

Foundations as a layered history

So the "foundations" considered here will start with a partial set of requirements for a fundamental construction-kit (FCK) provided by the physical universe with "more attractive Powers" unknown to Newton. and a partial discussion of how the mathematical potential of such a construction kit can be realised through biological evolution and its products.

I do not claim that the existence of the FCK suffices for the production of those mathematical developments: some chance conjunction of circumstances may be required that will not necessarily occur in all instances of the FCK. Not all galaxies will necessarily produce living organisms, and not all planets on which life develops will necessarily follow the type of mathematical trajectory that occurred on our planet, discussed in outline below.

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[7] `http://www.cs.bham.ac.uk/research/projects/cogaff/misc/construction-kits.html` "Construction-kits as explanations of possibilities." (Under construction!) There are hints in at least two of Turing's late papers that he thought chemical structures and processes were part of the answer to Question 2.

[8] Reported in `http://www.chemistryexplained.com/Ne-Nu/Newton-Isaac.html`

[9] Quoted in various internet sites, including:
`http://galileo.phys.virginia.edu/classes/252/atoms.html`

Foundations as the original construction kit

Some philosophers will request “deeper” foundations – e.g. a metaphysical theory explaining what makes possible the existence of a universe providing a construction kit with sufficiently rich generative power to support the evolutionary and other processes of mathematical development outlined below, or perhaps a mathematical theory characterising the set of possible universes in which such developments can occur. This may require the development of a new field of mathematics, studying different sorts of construction kit and their generative powers! If that happens it won’t be the first time a theory of the foundations of mathematics has proposed a new mathematical theory. No attempt is made here to meet that request explicitly, though some of the groundwork for an answer may be found here.

Requirements for the Fundamental Construction Kit (FCK)

Specifying the requirements will be a major long term project. Some of the requirements can be based on observation of physical forms of animals, plants and other forms of life (including microbes, and pre-biotic molecules). However, for present purposes empirical correctness is not a requirement: we are need to understand the range of possibilities rather than what actually occurred. Ideally mathematical analysis, as in (Turing, 1952), will demonstrate the relevance of any proposed features.

It will be useful if we can abstract from physical/chemical details in something like the way that the specification of a Turing machine abstracted from physical details. However, the turing machine specification did not mention energy requirements or the processes of construction, whereas in the context of explaining what made biological evolution possible the roles of energy and of assembly processes are essential to specification of any proposed FCK. A model partial specification is in the “Chemoton”theory of (Ganti, 2003).

Here are a few (incomplete) additional remarks on the requirements.

1. The FCK must suffice not only for the construction of all the types of living things that can occur, but also the physical environments containing non-living things, e.g. water, mud, sand, rocks, air, and many more, with which living things interact in multiple ways, e.g. treating objects made of physical materials for support, shelter, food, building materials, routes, obstacles, tools, protection against physical dangers and against predators, competitors, etc.

2. It must also allow energy and materials to be stored, transferred, and used in enabling a wide variety of physical processes some initiated by organisms (e.g. growth, repair, movement, reproduction) others initiated in the environment providing opportunities (e.g. winds, rivers and sloping ground, for transport) and dangers (e.g. tides, floods, volcanoes, strong winds, lightning, etc.).

3. It must provide a wide variety of mechanisms that can play different roles in biological uses of information, including various types of sensing (internal and external, local and remote)

Abstract for a possible future progress report on Foundations of Mathematics

Claim: there are different foundations for different mathematical phenomena and some aspects of mathematics as we know it have biological foundations that have not yet been investigated. Those, in turn, have foundations in the chemical construction kits available since before the dawn of life.

The study of biological/evolutionary foundations of mathematics (BEFM) is an approach to understanding the nature of mathematics that addresses question close to those raised by Kant in (Kant, 1781) about how mathematical knowledge of non-analytic necessary truths is possible, by

Turing's work on Turing machines and computability (Turing, 1936), and related 20th Century work on foundations of mathematics and automated theorem proving,[10] work by Piaget and his followers on mathematical aspects of cognitive development (Piaget, 1981, 1983; Sauvy & Sauvy, 1974), and most recently by my reading Turing's 1952 paper on morphogenesis (Turing, 1952), hinting at what he might have done if he had lived more than two years longer, as suggested in (Sloman, 2013b).

- *Evolution of organisms (implicitly) using mathematical structures in their environment*
  (e.g. homeostatic control mechanisms based on negative feedback; avian flight control mechanisms managing instability; syntax-based descriptions of environmental structures and processes, e.g. "A is between B and C"),

- *Evolution of mathematical abilities to notice and reason about those structures, including inferring properties and relations, and constructing explanations based on understanding mathematical structures of space and information flow*
  ("Why do I see more of the cave as I move closer to the entrance?", "If I move three paces then five paces forward where will I be?")

- *Evolution of mathematical abilities to explore possible extensions of those structures and alternatives to those structures?*
  (E.g. adding length and other metrics to systems of comparison: from "longer" to "three paces long", or thinking of lengths of curves, areas of irregular shapes, volume of an egg... before the development of differential and integral calculus)

- *Evolution of meta-cognitive meta-mathematical abilities to reflect on those reasoning abilities?*
  (Would the answer be different in another village? On a higher mountain? When the wind is blowing? Why not? What makes P *necessarily* true? Where do constraints come from? (Kant))

- *Evolution of meta-meta- abilities to reflect on possible alternatives to those reasoning and exploration abilities?*
  (Attempting to organise results into an ordered derivational system with explicit axioms. Attempting to find syntactic criteria for valid inference. Attempting to find indisputable starting points. Engaging in foundational debates: Can arithmetic be reduced to logic? Can geometry be reduced to arithmetic? Did Hilbert's axiomatisation of geometry (Hilbert, 2005) change the subject, as suggested by Frege? Did Frege's logicisation of arithmetic change the subject? (Frege, 1950)?)

- *Products of evolution (e.g. humans) discovering how to produce working replicas of some products of evolution*
  (proof checkers, automated reasoners, automated mathematical discoverers...)
  But why have we not yet been able to mechanise some competences,
  e.g. topological reasoning about continuous deformation of closed curves?
  `http://www.cs.bham.ac.uk/research/projects/cogaff/misc/torus.html`

Some universes (e.g. one made of Newtonian point masses) could not support reasoning mechanisms. A key feature of ours is *chemistry*, making possible an enormous variety of stable structures of varying size and complexity able to store usable energy, create motors, create sensors, create enclosing and supporting structures, create self-moving machines, and create *stable* but *changeable* information structures (using catalysis, thanks to quantum mechanics). (Ganti 2003). Turing machines are more restricted, unable to support mixed discrete and continuous operations.

That leads to a host of new questions: if X is possible what mathematical properties of the *processes of biological evolution* make X possible ... e.g. make possible mechanisms able to discover Euclidean geometry? Was Kant on the right track? Was it really pure logic somehow disguised?

[10] `http://en.wikipedia.org/wiki/Automated_theorem_proving`

Something else? (J.S.Mill? Wittgenstein? ...?) If it was logic, what made organisms able to do logical reasoning?

This leads to further questions: e.g. what new possibilities are enabled by mathematical properties of the *products of biological evolution*? Products include information-based control mechanisms of many kinds. What mathematical properties of various information structures and information-using mechanisms make possible various forms of perception, learning, reasoning, deliberating, acting, communicating, ... and mathematical discovery?

Example: what biological mechanisms made it possible for our ancestors to discover bits of geometry that led to the production of Euclid's elements and many discoveries in arithmetic–long before the development of formal proof methods or attempts to base mathematics on some well defined foundations. I think those early non-empirical mathematical modes of reasoning have never been fully understood. Euclid's *Elements* clearly reported and to some extent organised mathematical discoveries, not empirical discoveries (even if they started as empirical). Could the abilities of ancient mathematicians be connected with the abilities to perceive and reason about affordances in the environment – what is and is not possible, and why? (J.J.Gibson)

Those modes of perception and reasoning must in part be products of biological evolution that served needs of some intelligent animals, e.g. the need to acquire and manipulate information about what is and is not possible in an individual's environment (e.g. in a spatial configuration where some things move and others do not), and to discover consequences of realising some of the possibilities (e.g. how they would alter subsequent possibilities and impossibilities).

Example: what biological mechanisms made it possible for Descartes to notice the structural correspondences between parts of Euclidean geometry and sets of numbers, sets of pairs of numbers, sets of triples of numbers, and equations relating numbers – a mathematical discovery linking two domains? (No current robot could do that. What needs to be added?)

What mechanisms made it possible for Newton to use Descartes' results to discover and prove things about previously unnoticed aspects of the physical world, expressible in a new mathematical formalism? Abilities to invent, see and use syntactic structures are also mathematical competences.

And later on: what features of the forms of meta-cognitive information processing in humans make it possible for some of them to notice and begin to investigate systematically possible *alternatives* to the mathematical structures discovered and thought about up to any particular time – including non-Euclidean geometries, non-standard logics, non-standard arithmetics, different transfinite structures,...

[All this presupposes a non-standard theory of the semantics of modality, discussed elsewhere.]

Some aspects of abilities that underlie adult mathematical competences seem to play a role in various kinds of animal intelligence and pre-verbal human intelligence (We need examples of toddler discoveries, as well as examples from other animals, e.g. squirrels, weaver birds, elephants...[11]

Example (toddler?) theorem: If a shoe-lace goes through a hole in the shoe you can extricate it by pulling one end or the other end, but not both ends simultaneously – even if the lace is stretchable. Why not? Why is it a mistake to try to put your shirt on by pushing a hand into a cuff, and pulling the sleeve up the arm? Your answers need not depend on statistical evidence from failed attempts, because you can reason mathematically about everyday things, even unwittingly. It's very hard to axiomatise such knowledge, and to justify required premises/axioms/inference rules?

Examples of spatial intelligence in young humans and other animals suggest that biological evolution (blindly) "discovered" and made effective use of mathematical structures in both the

[11] `http://www.cs.bham.ac.uk/research/projects/cogaff/misc/toddler-theorems.html`

environments in which organisms evolved, and also in the space of possible biological information-processing mechanisms that we (human mathematicians, philosophers, computer scientists) have not yet understood, and which may provide new answers to old questions about the nature and scope of mathematical knowledge and how it differs from other kinds.

Those (object)-mathematical abilities can be present in some animals and young humans without the meta-cognitive abilities required to think about the discoveries and the reasoning processes, or to communicate discoveries to others, or give reasons. What has to change?

The meta-cognitive abilities are clearly not available at birth in humans, and seem to depend on later development of brain mechanisms that for important biological reasons are delayed in some intelligent species. Kant: "...faculty of knowledge ... awakened into action...", i.e. there is unreflective and (later) reflective mathematical learning.

Are foundations of mathematics and foundations of meta-mathematics necessarily different?

This may eventually provide new support for a modern variant of Kant's ideas: mathematical knowledge is synthetic and *a priori*, non-empirical, but neither innate nor infallibly derived (Lakatos 1976), and mathematical truths are necessary but not a subspecies of logical truths.

There are many unanswered questions about mathematical discoveries (e.g. discoveries about equivalence classes of curves on a planar, spherical or toroidal surface) that are close to human common-sense reasoning about spatial structures and processes but beyond the scope of current automated theorem provers.[12] Can that be fixed without radical innovation? Brain science has shed no light on any of this.

New advances in AI are needed to model these processes, before we can replicate the human mathematical capabilities in robots. What sorts of advances?

When we have examples of *working* explanatory models that may help neuroscientists decide what to look for, instead of blindly seeking correlations, as now.

It may also turn out that there are forms of computation (e.g. perhaps, as Turing hinted in 1950, chemistry-based computation, with its mixture of discrete and continuous processes) that provide a different space of mathematical mechanisms from Turing-equivalent mechanisms. Turing was thinking of chemical processes before he died (1952). He had a deep interest in how living things worked.

I think there's a rich space of mathematical structures waiting to be explored concerned with such evolutionary and developmental processes, which ultimately depend on the mathematical structures and processes supported by physics and chemistry.

We need to understand kinds of construction kit that are capable of supporting not only the changes in shape, physical and chemical processes, forms of reproduction, types of metabolism, etc. but also the constantly growing variety of forms of *information processing* from the simplest conditional switches (chemotaxis) and feedback control mechanisms (homeostasis) through to advanced mathematical, scientific, and philosophical reasoning and discovery produced by evolution.[13]

There are connections with Jean Piaget's developmental psychology(Piaget, 1981, 1983) and the work of neurodevelopmental psychologist Annette Karmiloff-Smith, e.g. her ideas about representational redescription (1992), and probably others I've not yet identified.

We need to attract clever, young, researchers from several disciplines to contribute to this type of research on foundations of mathematics. We may need new forms of education.

[12] `http://www.cs.bham.ac.uk/research/projects/cogaff/misc/torus.html`

[13] For work in progress, still at an early stage, see `http://www.cs.bham.ac.uk/research/projects/cogaff/misc/construction-kits.html`

(Ida & Fleuriot, 2012) (Karmiloff-Smith, 1992) (Lakatos, 1976) (McCarthy & Hayes, 1969) (Mueller, 1969)

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