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Deforming triangles and the Apollonius problem

(A surprisingly complex outgrowth of a simple problem.))

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With grateful thanks for help from
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I am also very grateful to Alexander Bogomolny's extraordinarily useful, freely available, web site, which Diana and I both used. It is full of mathematical miscellanea, including some wonderful applets:
https://www.cut-the-knot.org/front.shtml
https://www.cut-the-knot.org/wanted.shtml

N.B. Aaron Sloman is responsible for any errors in this document.

Installed 17 Nov 2017
Updated:20 Nov 2017; 4 Dec 2017; 29 Dec 2017; 11 Jan 2018
This paper is
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/apollonius.html
A PDF version may be added.

This is part of the Meta-Morphogenesis project:
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/meta-morphogenesis.html
A partial index of discussion notes on this web site is in
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/AREADME.html

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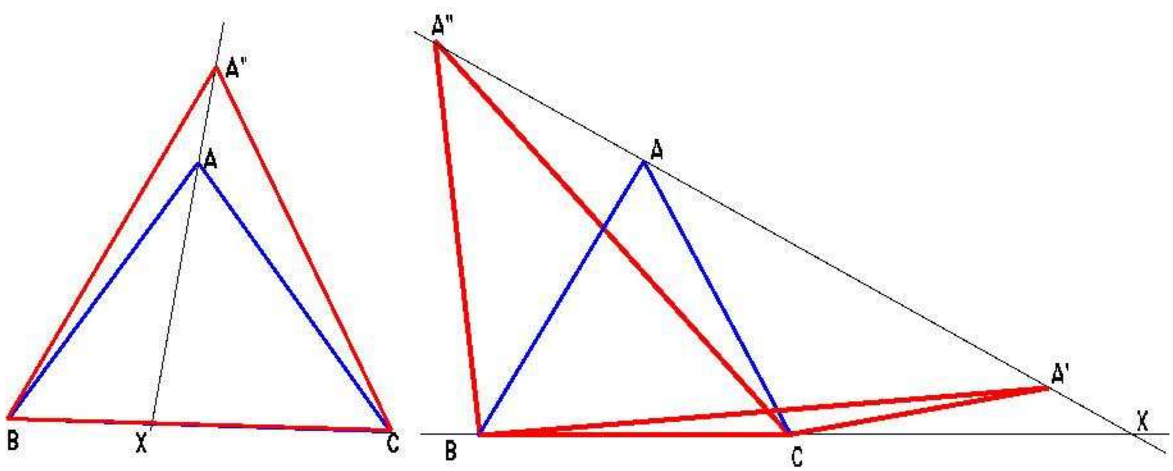
Background

This document is a "side-shoot" of a document on spatial cognition, which is part of a collection exploring mathematical consciousness and the mechanisms on which it depends, using examples related to ancient discoveries in geometry and topology. I don't think there is anything in current neuroscience, or in AI/Robotics, capable of explaining or modelling the discoveries and forms of reasoning of ancient mathematicians presented below, or the intelligence of pre-verbal toddlers, nest building birds, squirrels, etc., and common non-metrical ways of reasoning about affordances, that I think are closely related to such ancient mathematical reasoning in unobvious ways, mentioned below, and in Sloman(2007-14)).

QUESTION

> What happens to the size of angle A of a triangle ABC as A moves further from from BC along a straight line cutting line BC between B and C, as depicted below on the left?
>
> What difference does it make if A is moved along a line cutting BC outside the triangle, as depicted below on the right?

Two ways to deform a triangle



Compare deforming triangle ABC by moving vertex A along a straight line going between

B and C (on left) and moving vertex A along a line that intersects BC outside the triangle (right). In both cases X is the intersection between line BC and the path of motion of vertex A.

The problem in the first form (on left) was discussed briefly in a short presentation at the ICCM Conference in July 2017, and contrasted with the forms of reasoning required for doing logic.

Adult academics from several disciplines, when asked the question about the figure on the left, seem to find the answer obvious, saying that as the vertex moves further from BC the angle must get smaller. They were able to do this even even if the problem was presented without showing a picture, e.g. if I merely described the triangle and type of change in words, possibly using gestures to indicate what I was referring to. (None of those I have asked had previously considered the question.)

It is not at all clear what form of reasoning they are using -- nor what their brains are doing. They don't seem to be aware that in answering the question they have identified a relationship between two mathematical continua: the continuum of locations of the moving vertex on the line through the triangle, and the continuum of angle sizes for the vertex as it moves. They find it obvious that the two must be correlated: e.g. the angle size cannot start increasing after the vertex A has moved beyond a certain distance.

(The figure on the right, where the intersection point X is not between B and C, is more complex and will be discussed below.)

I discussed the first case (with X between B and C) in more detail in an invited talk for an IJCAI workshop in August 2017 in Melbourne. The talk was presented remotely:

Why can't (current) machines reason like Euclid or even human toddlers?
(And many other intelligent animals)
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/ijcai-2017-cog.html

That website includes a link to the online video presentation, discussing differences between discoveries in geometry and in logic, using the stretched triangle example and simple examples in propositional logic. It includes examples of 3D topological reasoning in a 17.5 month old human toddler and weaver birds.

More detail was later added in a document prepared for a seminar in my own department on 29th September 2017:
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/deform-triangle.html
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/deform-triangle.pdf

A previously unnoticed problem

On that occasion one of our students, Auke Booi, pointed out a complication I had not noticed. If the vertex A starts close to the extended line BC outside the triangle, as in the figure on the right, above, then the angle size must increase for a while as the vertex moves away from line BC, before it starts decreasing.

In the figure on the right, it is obvious that if stretching starts at location A then the angle must get smaller, but, as Auke pointed out, if the vertex starts near location X, in the right hand figure, then the size of the angle will be small. As it moves further from X the size will increase at first. Later it starts decreasing.

At what location on the line on the right is the size of the vertex largest? The answer is not at all obvious.

Later, in a talk at the School of Computing, Leeds University, on 3rd Nov 2017, I again talked about the two stretched triangle processes, and invited the audience to think about what must happen to the angle size at a vertex, if the vertex is moved along a straight line that crosses the line through B and C, outside the triangle, as on the right above.

It is obvious that the size of the angle must first increase then decrease. But it is not at all obvious where the size reaches a maximum. (I have no idea whether anyone else has ever previously formulated this problem.) It is fairly typical of geometrical and topological reasoning that there are short transitions from problems and solutions that are easy to understand to problems that are much more difficult, raising questions about the cognitive mechanisms required for the easy and the difficult cases.

The following morning, at the PTAI17 conference, also in Leeds, Diana Sofronieva, a philosophy student who was helping with organisation, told me that after attending my talk she had thought about, and solved, the problem about the location of the maximum angle size. Her solution turned out to be surprisingly complex: it requires a lot more prior knowledge of results in Euclidean geometry and many intermediate steps between formulation of the problem and derivation of the solution: the location where the angle is maximal. Perhaps it would not be surprising for a mathematician deeply familiar with Euclidean geometry.

I offer the solution below, not as a contribution to mathematics (though for all I know, my question has never previously been asked and Diana’s discovery and proof of the answer have never previously been noticed), but as yet another example in a growing collection of forms of mathematical reasoning whose mechanisms don’t seem to be recognized in current neuroscience or psychology, and are missing from the (current) arsenal of AI reasoning mechanisms.

Giving a precise specification of the construction of an answer to the question "Where is the angle size largest?", applicable to all planar triangles, and proving its correctness, turned out to depend on a construction and proof, known to ancient mathematicians as Apollonius’ problem (which has several different forms). E.g. see
https://www.cut-the-knot.org/pythagoras/Apollonius.shtml

Using Diana’s solution, and the useful links she supplied, this document describes the construction process, summarises the proof of correctness, and provides links to further online information about Apollonius’ construction. Later work will discuss implications for the Super-Turing Membrane machine mentioned below, and discussed in another document.

Need for a new kind of (virtual?/physical?) machine

It is interesting that what appear to be simple and commonplace abilities to reason and ask questions about changing configurations of simple shapes (e.g. straight lines, triangles, and circles) can generate such a tangled network of construction processes with associated reasoning about the properties of resulting diagrams. What kinds of brain mechanism could explain the processes of discovery and reasoning that occur? How are they related to ordinary mechanisms of perception and reasoning, e.g. about affordances in the environment? How did all of this apparatus evolve? How does it develop in individual brains? Can computers/robots be given similar powers of mathematical discovery? I don’t think anyone knows the answers at present.

I have begun to use the label "Super-Turing Membrane Machine" to refer to the kind of information processing machinery that seems to be required to enable a brain or a future robot to make such geometrical and topological discoveries, find proofs, and understand why they work, e.g. how they demonstrate possibility, impossibility and necessity. This is completely different from discovering statistical regularities e.g. by sampling large numbers of images. The specification of such a new virtual machine is still incomplete, but is being constructed piecemeal, starting here:

> http://www.cs.bham.ac.uk/research/projects/cogaff/misc/super-turing-geom.html
> A Super-Turing Membrane Machine for Geometers
> (Also for human toddlers, and other intelligent animals.)

The original (less mathematically complex) presentation of the problem of how the size of angle varies as the vertex moves is available here:
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/deform-triangle.html

These are all part of the Turing-inspired Meta-Morphogenesis project

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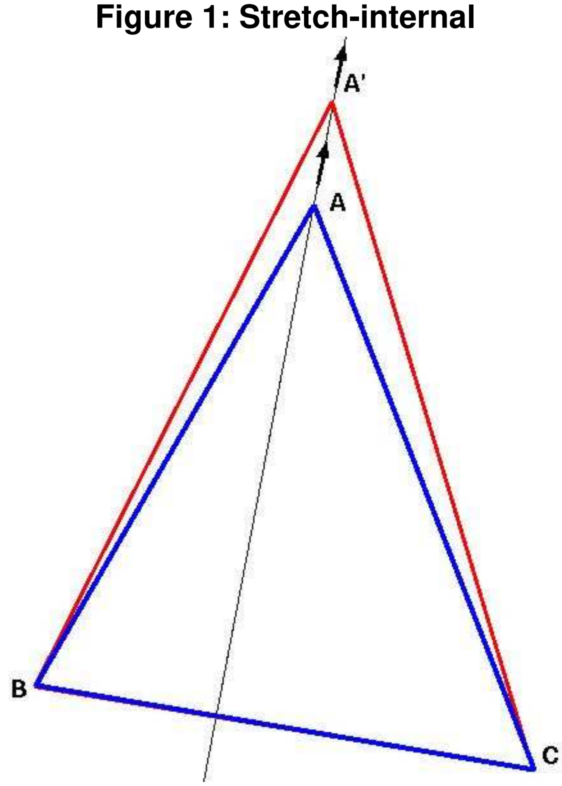
What needs to be explained?

There have been many attempts to understand brain mechanisms supporting mathematical abilities. Unfortunately, most start from an inadequate characterisation of what needs to be explained (a topic for another time). In a talk at the International Conference on Cognitive Modelling (ICCM) in July 2017, I decided to use a new problem that, as far as I know, does not figure in mathematical text books, namely

> "If ABC is a triangle of any shape, and the vertex A is moved in a straight line away from the opposite side BC, what happens to the size of the angle at A?"

Figure 1: Stretch-internal



Possible forms of reasoning are discussed in the IJCAI 2017 workshop presentation.
The example is also discussed in the video for the IJCAI talk: [VIDEO LINK]

Everyone I talked to seemed to find it obvious that the angle must steadily decrease in size. This example (like many others, e.g. available in http://www.cs.bham.ac.uk/research/projects/cogaff/misc/impossible.html) can be used to motivate questions about the nature of the perception and reasoning mechanisms that allow such discoveries to be made and to be seen to be necessary truths: e.g. if the line on which A moves passes between B and C, i.e. crosses the opposite side of the triangle, then the angle at A must decrease as A recedes from BC.

These examples, like many others on the CogAff web site, raise interesting questions about the cognitive machinery required identify necessary truths and impossibilities. Immanuel Kant raised such questions in Kant(1781).

Moreover, insofar as people notice that the angle necessarily (eventually) grows smaller as the distance from the opposite side increases, and without having done any experiments or collected statistics (except perhaps implicitly and unwittingly over sub-ranges of shape change), they cannot be reporting an empirical generalisation. Instead they are conscious that there is a necessary connection between the change of length and the change of angle.

Consciousness of such necessities, and corresponding impossibilities, is a characteristic feature of mathematical discoveries (as Kant pointed out), and is utterly different from empirical (Humean) reasoning by generalisation from examples, even if mathematically sophisticated statistical reasoning is used.

These aspects of mathematical discovery (in geometry, topology and arithmetic) are ignored in most of the empirical research on mathematical competences by psychologists and neuroscientists. Piaget was one of the few exceptions Piaget (1981,1983).

The perception of necessity and impossibility also seems to be ignored in the vast majority of philosophical discussions of consciousness and in AI. I don't know of any AI theory, whether concerned with vision, learning or mathematical reasoning, that even attempts to explain how such ancient mathematical discoveries concerning impossibilities and necessities are made, and recognised as such. The best available AI theorem provers generally use logical and algebraic forms of reasoning that were not available to the ancient mathematicians at or before the time of Euclid. Neither are those machines aware of the contrast between empirical and mathematical discoveries.

There are many online graphical tools for manipulating and exploring mathematical problems, but none of them (as far as I know) includes a discovery mechanism capable of learning what human mathematicians learn by playing with the tools.

Any theory of consciousness that does not account for mathematical consciousness (including consciousness of mathematical necessities and impossibilities) must be inadequate: which applies to almost everything I have read about consciousness, except in Kant's work, which states the problem but does not solve it.

(Several mathematical models of consciousness have been proposed, e.g. based on mathematical properties of dynamical systems that are claimed to model brain mechanisms, but the examples I have encountered do not explain, or even describe, mathematical consciousness! I have tried to explain this to some of their proponents -- so far without success.)

How are circles relevant?
Added/updated 8-9 Nov 2017

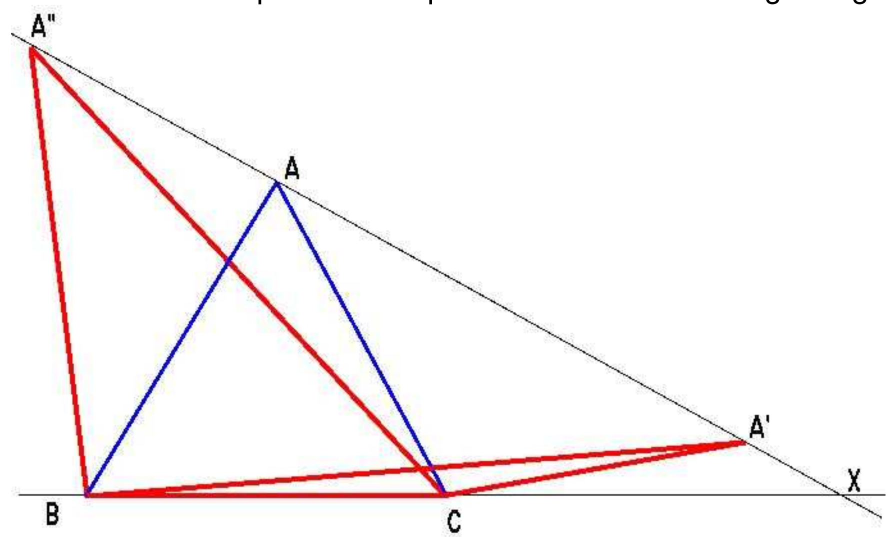
At a seminar in the School of Computing at Leeds University on 3rd Nov 2017 I presented the question posed in Figure 1 above as an example of geometric reasoning that is different from logical or arithmetical reasoning.

The question was then generalised by allowing the line of motion of the vertex A to cross the line through BC outside the triangle. Auke Booij, a student at Birmingham, had previously pointed out that if the vertex A moves along a line that crosses the base of the triangle outside the triangle, e.g. at X in Figure 2 below, then for some parts of the motion along the line the size of the angle at the vertex A starting near X will increase as the vertex moves further from the line BC, though it obviously decreases beyond some point on the line.

This raises the question: for which position of point A along the line XA" is the angle BAC largest? Auke had pointed out that the angle could start small, increase, then reduce, as is obvious in Fig 2. What sort of brain can find this obvious? What sort of reasoning can identify, in a generic form, the location where the size of the angle at the moving vertex is maximal. Could there be more than one peak (for a given straight line XA")?

Figure 2. External intersect

If A moves between point X and point A" where is the angle largest?



A solution involving circles

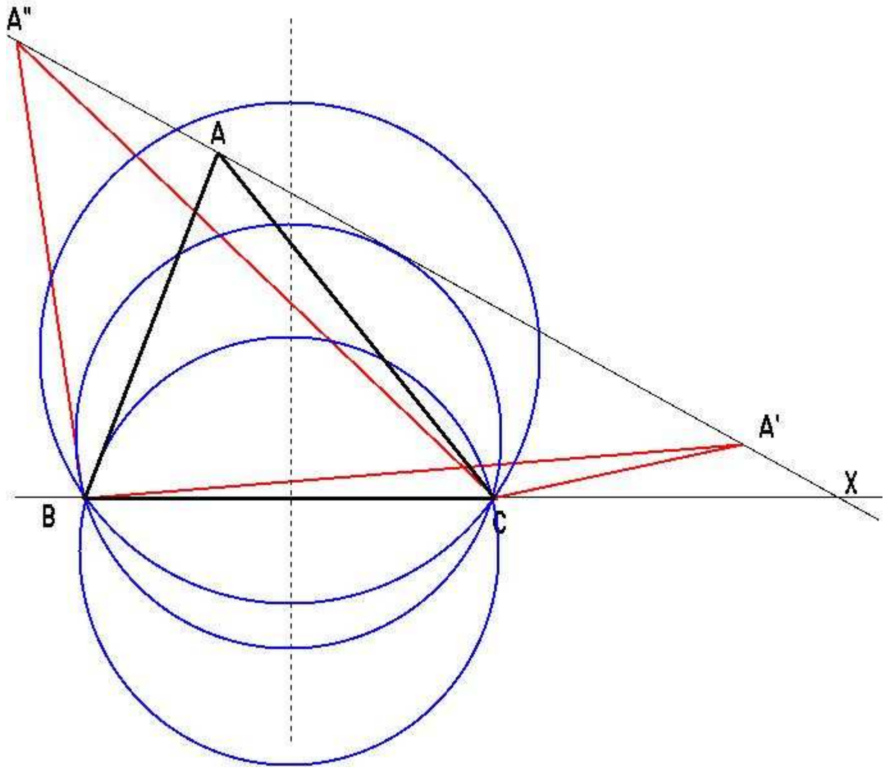
Diana's solution to the problem of finding the location at which angle A is maximal makes use of a mixture of fairly elementary geometry and a fairly complex ancient construction named after Apollonius, explained below. I had never previously encountered the problem. It has a surprisingly complex solution. The proof that it is a solution uses elementary geometry plus the Power of a point theorem, also explained below.

So the sliding vertex problem, which is very easy to understand, leads very quickly to some complex and obscure geometrical reasoning. This happens often in Euclidean geometry, and in other areas of mathematics!

My presentation below is based on Diana's solution. I altered her diagram (Figure 6 below, repeated later as Figure 6a for easy reference in the PDF version of this file) and her wording, to be consistent with my earlier diagrams. I apologise for any errors introduced. First I'll introduce a collection of circles, relevant to the solution to the problem. Figure 4 below will use one of the circles to explain why the solution depends on a circle that has line XA" as a tangent.

Figure 3. Circles

Circles through B and C, one below line XA", one touching it and one intersecting it"



There are no circles visible in the earlier diagrams (before Fig. 3). However, the Euclidean plane allows infinitely many circles to exist. In particular, there are infinitely many different circles passing through the two points B and C, the bottom vertices of triangle ABC in Fig. 3 above. That's because all points on the perpendicular bisector of BC (depicted by the vertical dashed line in the figure), are equidistant from B and C.

Therefore every point on that line determines a unique circle passing through points B and C.

It turns out that the answer to the question where the angle A is maximal depends on such circles and their tangents! (Unless someone can find an alternative construction of the location of the maximal angle size?)

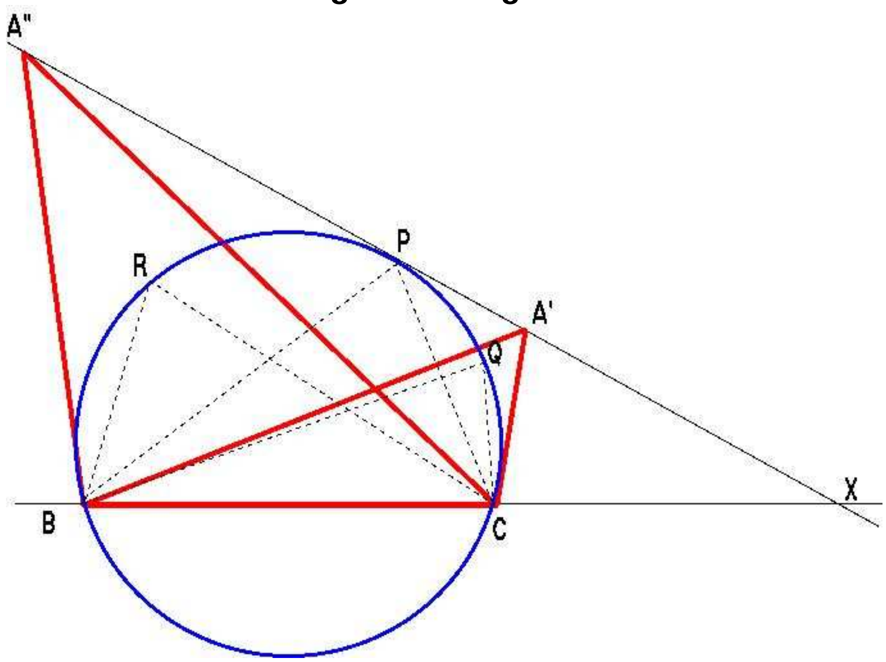
Three example circles are the blue circles shown in Fig. 3. Each passes through points B and C, and because the centre of each circle must be the same distance from B and from C, the centres must all be on the perpendicular bisector of the line BC.

Contemplating Fig. 3 should convince you that there are infinitely many circles that pass through the points BC and are entirely below the line from X to A", and share no point with that line, as illustrated by the lowest blue circle in Fig. 3.

Likewise there are infinitely many circles through B and C that pass above the line XA", cutting it in two different points, as illustrated by the highest blue circle.

However, as the diameter of a circle through B and C shrinks or expands (while the centre moves up or down the perpendicular bisector of B and C) there there must be exactly one circle through points B and C that touches the line between X and A": i.e. that line is tangent to the circle at a point indicated by P in Fig. 4 below.

Figure 4. Tangent1



In Fig. 4 (derived from the diagram Diana produced), given the line XA" consider the circle passing through B and C for which XA" is a tangent, at a point P. In that case, the maximum angle subtended by BC for points on XA" must be the angle at P. Why? The answer is given below.

The uniqueness of the tangent point between X and A" is not obvious but is implied by the construction and proof below. However, if circles are allowed with centres below the line BC, then another such tangent point exists, as explained later.

The existence of such a circle with line XA" as tangent is "intuitively" obvious, but needs a proof, discussed below. Its uniqueness is easy to see, given that the centre of any circle through B and C must lie on the dashed line, the perpendicular bisector of BC.

However, looking back at Figure 3, if we consider circles with centers on the perpendicular bisector of BC, but below the line BC, the portions of those circles passing through B and C will be mostly below the line BC. As the centre moves further down the radius increases, the circle becomes "flatter", and we'll see later that eventually the circle will again meet the line XA" but below the line BC, i.e. to the right of X (at point K in Figure 6 below.

Why is the angle at the tangent point maximal?

Consider the blue circle passing through B and C, and touching line XA" at point P, the tangent point. We shall show that the angle BPC is larger than any other angle subtended by the line BC at any point on the tangent line XA". E.g. the angle subtended at point A' must be smaller than the angle at P. Why?

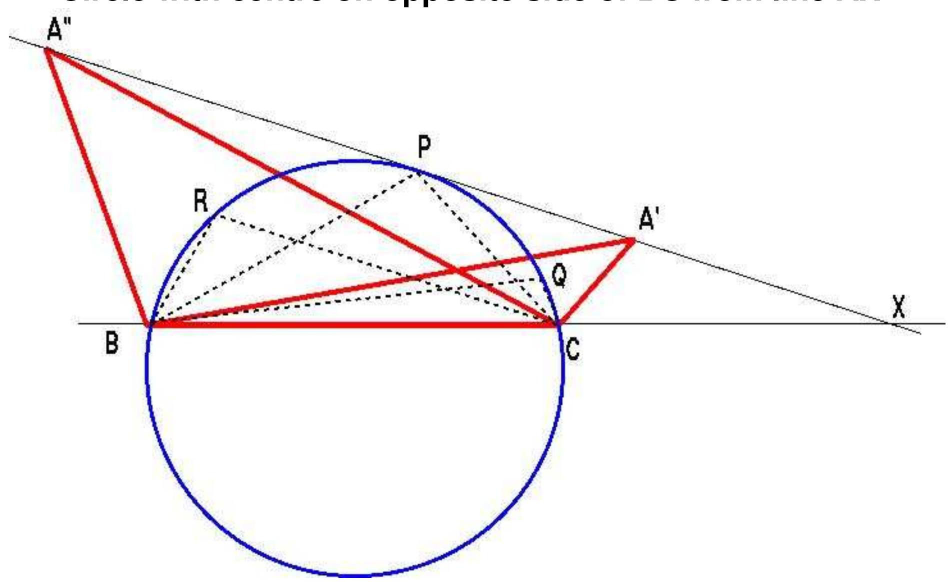
If we choose any other point Q on the circle, within the triangle BA'C, the angles BPC and BQC will be the same, because each is half the angle subtended at the centre of the circle by the chord BC. (This is a standard theorem of Euclidean geometry.)

Likewise angle BRC is the same as angle BPC. So angles P, Q and R must all be the same size.

However, angle BA'C must be smaller than angle Q, because that's an example like Figure 1. For the same reason BA"C must be smaller than angle R. Therefore both angles BA'C and BA"C are smaller than angle BPC. Since A' and A" are arbitrarily chosen points on either side of P, the angle subtended by BC at any point on the line through A' and A" must be largest at P, the tangent point.

So, in order to discover where the angle BPC, with P a point on the line XA" is maximal, it suffices to discover the circle through B and C for which the line is a tangent.

Figure 5. Tangent2:
Circle with centre on opposite side of BC from line AX"



What happens if there is no circle with centre above BC that meets the line as a tangent? In that case, as illustrated in Fig. 5, there must be a circle with centre on the opposite side of BC, still on the perpendicular bisector, which passes through B and C and has the line as a tangent. (How could we build a machine that finds that obvious?)

We can use the same reasoning as in Fig. 4 to show that the angle subtended at BC is largest at the tangent point P in Fig 5. At any other point on the line XA" the angle will be smaller because the point is outside the circle (as in Fig. 4).

Later we'll see that for any line *l* cutting the line through B and C outside the segment BC, there are two circles that have tangent points on *l*, though their locations may not be obvious at first..

Apollonius' Construction
Almost all of this section is based on help from Diana Sofronieva,

How can we construct a circle that passes through B and C and has the line XA" as tangent? This requires the solution to Apollonius' problem. The problem attracted the attention of several outstanding ancient and modern mathematicians. There are several variants of the problem that will not be discussed here: they are easily found online.

I have slightly modified the diagram Diana sent for consistency with my previous diagrams.

In Figure 4 it was assumed that given line segment BC, and the line XA" we could construct a circle through ends B and C for which XA" is a tangent line. We later found that depending on the relationship between points B and C, and the line XA" the solution will use either a circle with centre above or on line BC, as in Fig. 4 or a circle with centre below line BC as in Fig. 5.

We need to specify a Euclidean construction (i.e. using only straight edge and a pair of compasses) that provably uniquely specifies the location of the desired circle (to which XA" is a tangent, and which passes through points B and C), if such a circle exists, or which allows us to find all circles satisfying the condition, if more than one does. (There is no such circle if XA" intersects the line BC between B and C.)

This can be done, for example by a construction that specifies exactly where the centre of such a circle is and the exact length of its radius.

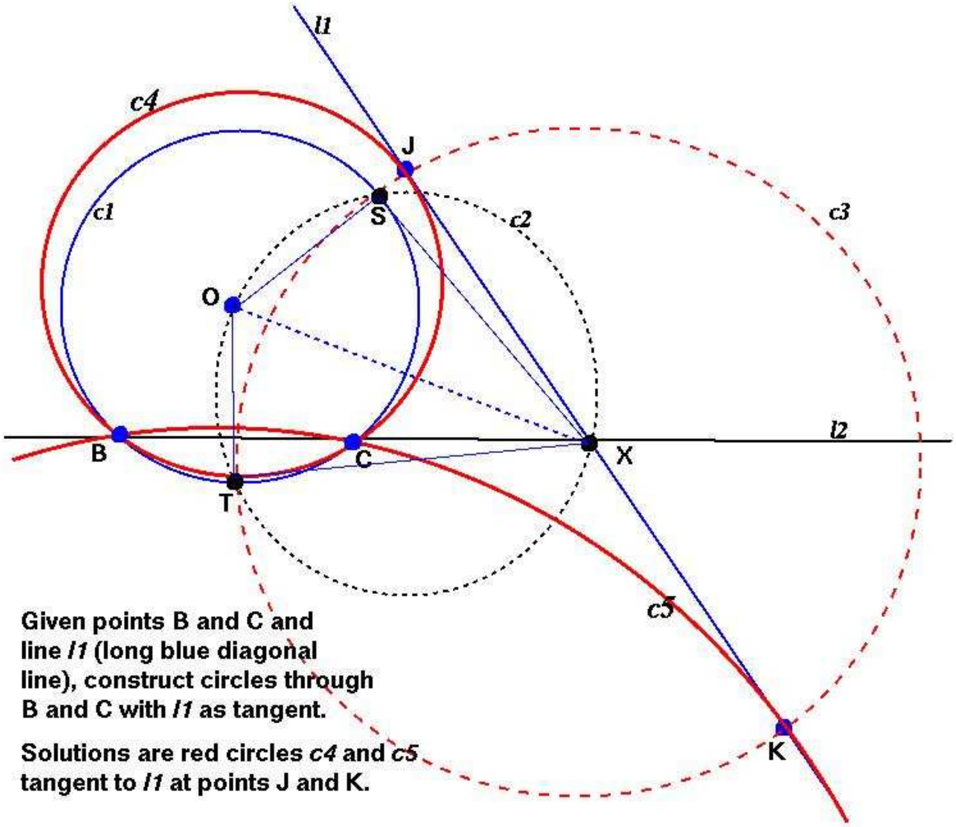
An intuitive interactive presentation of the problem and a (very unobvious) proof that there is a Euclidean construction that uniquely identifies the circle were found online by Diana here:
https://www.cut-the-knot.org/Curriculum/Geometry/GeoGebra/PPL-Simple.shtml

There are two closely related discussions on the same web site:
https://www.cut-the-knot.org/Curriculum/Geometry/GeoGebra/PPL-Simple.shtml
including this historical note giving a list of variants of the problem and their proofs:
https://www.cut-the-knot.org/pythagoras/Apollonius.shtml

Thanks to help from Diana Sofronieva, I shall now first present the construction that produces the two tangent points at which angle-sizes are maxima (one above the line BC and one below) and then a proof, using the same diagram, that the construction does what is required.

I have altered Diana's presentation to be consistent with the diagrams above and below.

Figure 6: Towards a solution of Apollonius' problem



The above version was designed to be consistent with the earlier diagrams, with points B, C, and the line XY given at the start. A compressed specification of the construction follows:

1. Let the line through B and C (black, *l2*) intersect *l1*(blue line) at point X.
2. Draw any circle, e.g. *c1* (blue), that passes through both B and C and does not cross l1. The centre of *c1* is O (blue dot).
3. Draw another circle, *c2*, (black dashed) with diameter OX.
4. Let *c1* and *c2* intersect at points T and S.

    Compare triangles OSX and OTX
    OS and OT are radii of circle *c1* therefore equal.
    Angle OSX and OTX must be right angles because they are on circle c2 with diameter OX
    Therefore triangles OTX and OSX are congruent but mirror images.
    Therefore XT = XS

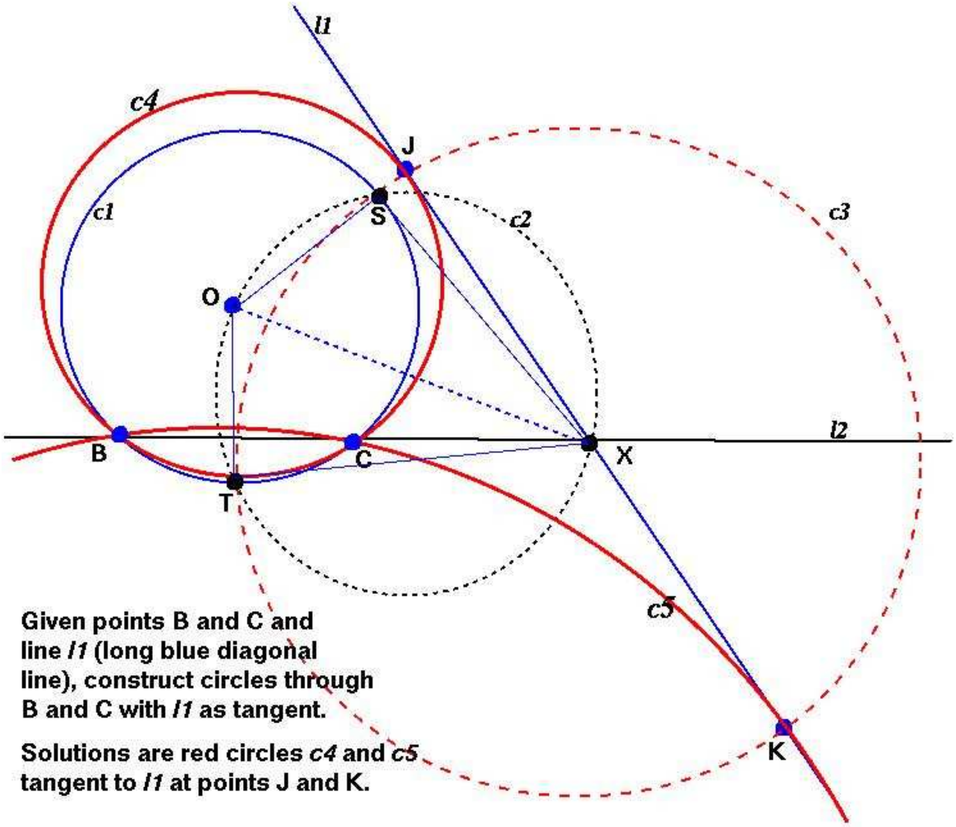
5. Draw a third circle, *c3* (red dashed), with a centre X and radius XT = XS
6. Let circle *c3* intersect line *l1* at points J and K (blue dots).
7. Given three non-collinear points there is a standard way to find the centre of a circle passing through all of them, by taking two at a time and drawing perpendicular bisectors. They must meet at the centre of the required circle. (The proof is easy!) So
    Draw a circle *c4* (red) passing through the three points B, C, and K.

Draw a circle *c5* (red) passing through the three points, B, C and J.
(Because it is so big, only a small part of the circle c5 is shown.)

The red circles (BCJ = c5) and (BCK = c4) are the required solutions, the circles through B and C meeting line *l1* at a tangent, at J and K respectively.

We now need to prove that the circles have line l1 as tangent, at points J and K. The proof will use the Power of a Point theorem, a standard theorem of Euclidean geometry, presented with proof here: https://www.cut-the-knot.org/pythagoras/PPower.shtml

Proof that red circle BCJ (c4) has line l1 as a tangent at point J:



Consider the red circle BCJ=c4.
[The proof for circle BCK=c5 will be exactly the same.]

1. Circle c4 passes through both B, and C by construction.
   So it remains to be proved that blue line l1 is a tangent of circle c4.
   Draw lines OS and SX, and lines OT and TX (blue lines)
2. Since OX is a diameter of *c2* (black dashed circle), angle OSX is a right angle (likewise OTX).
   therefore SX is a tangent to our initial circle c1 (blue) (likewise TX).
3. By the Power of a Point theorem, applied to our initial circle *c1*,
   BX.CX=XS**2 (i.e. "XS squared")
4. XS = XJ, since both are radii of *c3* (big black dashed circle).
5. Therefore BX.CX=XJ**2.
6. By the Power of a Point theorem again, it follows that XJ is tangential to the circle (BCJ = c4).
   Which is what we wanted to prove. I.e. we have found a circle through B and C which has line l1 as tangent, at J.
7. By a similar argument circle (BCK = c5) has line l1 as tangent at point K.
   (Exercise for the reader!)

The diagram from which I started the Apollonius construction and proof was based on this one,
https://www.cut-the-knot.org/Curriculum/Geometry/GeoGebra/PPL.shtml
on the Cut-the-knot site, found by Diana, and used in her explanation to me.

She also drew my attention to the bottom of the page, where it states:
"The solution is a pure application of the Power of a Point Theorem".

See also the following Cut-The-Knot page with an interactive GeoGebra applet showing how the tangent circles change as the configuration changes.
https://www.cut-the-knot.org/Curriculum/Geometry/GeoGebra/PPL-Simple.shtml
Points A and B correspond to my B and C, and the line m corresponds to my given line l1, meeting BC extended at X, in Figure 6.

Any errors in my diagram, or the commentary are entirely my responsibility!

Discussion: What sort of reasoning machine is needed?
[19 Nov 2017: WARNING! Incomplete draft:To be revised and extended]

It is a very important aspect of human intelligence that our ability to visualise lines and circles, and then consider combining them in a planar surface and deforming them in various ways, sometimes enables us (and perhaps other intelligent species reasoning about possible actions, such as nest building birds) to discover new possibilities and impossibilities.

Moreover, imagined, visualised, or drawn operations that are inherently fairly simple, can produce, when combined, extraordinary mathematical problems, solutions and proofs, including some that most people will find hard to follow, although they are not difficult for well educated mathematicians. What forms of brain development produce that enhanced, specialised new competence?

An important feature of the descriptions of structures and relationships, and the reasoning about them, above is an implicit feature: namely no specific lengths or angles (apart from perpendicularity) are mentioned in the subdivision into cases, although some pairs of lengths are said to be equal. It follows that the distinctions do not depend on the particular locations of the original points and lines in the diagram, only on their mutual relationships, e.g. perpendicularity, parallelism, being to the right or left of, and being nearer or further.

Because of that, what is demonstrated has "broad brush" scope: it applies directly not just to the particular figures presented, but to infinitely many different configurations of points, lines and circles satisfying the descriptions given.

In that sense, what is proved, as Kant noticed, has a kind of universality, and necessity, both completely absent from the kinds of facts supported by evidence presented in the empirical sciences, i.e. evidence for a probability distribution available from a statistical sample.

Moreover, the use of a diagram in mathematical reasoning does not depend on the diagram actually being an instance of what is under consideration: e.g. in my diagrams the lines are not infinitely thin, and they do not intersect in infinitely small points. Moreover, if drawn on paper, or on a blackboard, the lines may even obviously be slightly curved or wiggly, whereas what is being discussed involves only perfectly straight lines.

I suspect there is nothing in current neuroscience that explains how that is possible. Likewise current AI/Robotics.

All this implies that in ancient mathematical reasoning the diagrams did not have to be instances of what was being discussed. Instead they function as representations of the geometric structures and processes. (This point was also made, perhaps too obscurely, in Sloman(1971) and chapter 7 of Sloman 1978, inspired by Kant(1781).)

For example, if a triangle happens to be drawn with roughly equal sides that need not imply that its use is restricted to reasoning about equilateral triangles. All of this shows that perception of a diagram in a geometrical proof is a very sophisticated process, and that the reasoning processes do not depend merely on visual inspection of properties of the particular diagram. This is very different from the way in which large collections of images are used as evidence for generalisations in current machine learning systems.

In that respect, the use of examples in mathematics is totally different from the uses of observed examples in sciences like botany, zoology, chemistry and physics.

Nevertheless, I suggest that the fact that spatial (geometric and topological) relationships can often be identified and reasoned about without knowing precise locations, angles, distances, or orientations, is an important feature of everyday intelligent behaviour of non-mathematicians in spatial environments. Sloman(2007-14) presents examples where perception of partial orderings, and certain phase transitions without precise boundaries, can be useful in answering questions, formulating intentions and plans, and predicting possible failures of actions, leading to plan modification.

That kind of "common sense" spatial intelligence, shared with other intelligent species, is a precursor to the kinds of intelligence used by ancient mathematicians who clarified, made precise, and organised pre-existing intuitive discoveries about spatial structures and processes.

This fact could be the basis of far more intelligent robot behaviour than systems that depend on extracting probability distributions over large sets of precise metrical values.

Clarification of terminology to be added
metric space
partial orderings
phase transitions

This fact implies that much research in vision and robotics that assumes vision needs to provide information about actual locations, distances, orientations, etc, in order to be useful for intelligent action may be seriously mistaken, except in highly specialised contexts (e.g. jumping to the top of a high wall, or throwing a rock at coconut out of reach, where most of the motion is ballistic). The design of the "membrane" mechanism under investigation will need to be based on these requirements. For now I'll ignore the ballistic case, and consider only controlled action -- online not offline intelligence.

Not only is it a mistake to assert that precise geometrical information is needed in controlling actions, but attempts to derive such information from noisy, ambiguous, sensory data lead to unnecessarily complex probabilistic reasoning mechanisms (concerning sets of possible precise values) that are overkill for the task, and can prevent the kinds of understanding that lead to creative novel solutions.

Evolution seems to have provided better mathematical answers to some of the design questions than current robot designers and vision researchers. The "better" answers seem to use perceived geometrical and topological relationships, including partial orderings, rather than perceived metrical values (as conjectured in Sloman(2007-14) and Sloman et al.(2006)).

Another, more subtle difference is that the proposed methods include perception and representation of impossibility, possibility, and necessary connections (as Kant pointed out). These concepts are not related to statistical evidence. But there are unanswered questions about how those perceptual contents are represented and used, and the mechanisms by which they are derived from sensory data.

Related forms of geometric reasoning

Examples of reasoning about how deformation of triangles affects areas rather than angle sizes can be found in this discussion of how the area of a triangle changes as a vertex is moved: http://www.cs.bham.ac.uk/research/projects/cogaff/misc/triangle-theorem.html

I have no idea how brains could implement the sorts of mechanisms proposed here. Yet it is clear that at least some human brains can support the sorts of reasoning and evaluation/criticism of discoveries and conjectures here. Given the variety of species that clearly have kinds of spatial intelligence that allow them to make sensible choices in quite complex situations (especially nest building, but also caring for offspring), and the evidence that pre-verbal human toddlers can perceive structures, relationships and opportunities involving 3D topology and geometry (as demonstrated in this video

It may be that Turing's interest in chemistry-based morphogenesis shortly before his death was just the first step in a potentially very complex investigation of uses of chemistry to implement information processing mechanisms like the proposed non-turing membrane machine. (Still too ill-defined.)

Recapitulation

The September 2017 talk on triangle deformation, here, presented (partly conjectured) modes of perceiving and reasoning used by ancient mathematicians (e.g. Archimedes, Euclid, Zeno, etc.) that have so far not been replicated in AI systems, and which don't seem to be supported by neural network models, including deep learning models, because those mechanisms cannot discover, or even represent examples of impossibility or necessity, rightly stressed by Kant(1781) as intrinsic features of mathematical knowledge.

The ideas about mechanisms required for a brain, or AI system, to make the discoveries discussed here (and in related papers) are still very vague, and definitely incomplete, but the general idea is that instead of 1D, 2D, 3D, etc. regular arrays of numerical or symbolic values, or trees, graphs, and networks of linked discrete items, the membrane machine would allow information items to be "painted" into it, forming shapes of many kinds that can be deformed continuously in many different ways, including parallel deformations of the same or different items.

There is nothing new about that idea: many drawing/sketching software packages have been produced since the 1970s, e.g. to help architects, engineers and others with design tasks, to teach mathematics, or to act as playthings. Usually such packages are implemented in array-like data-structures, though in some cases the pictures are represented algorithmically and drawn as required (a simple example being a LOGO based drawing package of the sort proposed nearly 50 years ago for educational purposes -- a recent example being https://turtleacademy.com/).

Such drawing processes will typically often generate unintended new relationships -- e.g. because previously unrelated structures become neighbours, or mutually obstructive, as a result of such changes or deformations.

What will have to be newly designed is a collection of cognitive and meta-cognitive mechanisms able to inspect those painting and drawing and deforming process, notice some of their features, and thereby discover new possibilities and impossibilities.

This can be contrasted with the way in which a fairly simple digital computer can inspect possible combinations of truth values of a collections of boolean expressions, and deduce, for example, by exhaustive analysis that if these are **true**

```
P or Q    not-Q
```

Then **true** is the only possible value for

```
P
```

consistent with the definitions of the boolean operators **or** and **not** (as defined by the standard truth-tables).

The design of a machine capable of performing the required geometrical reasoning (without "cheating" by using logic and coordinate geometry to derive conclusions from logically formulated axioms) is discussed in the still highly sketchy requirements for a new type of machine.

Properties of the Super Turing membrane

(This section and the next section have been moved into a separate discussion of the conjectured "Super-Turing membrane machine"

Monitoring mechanisms

This section also moved.

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Note: A presentation for non-mathematicians of the main ideas in Turing's paper can be found in Ball, 2015.

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